Duality in LP

Associated with every LPP there always exists another LPP which is based upon the same data and giving the same solution .The original problem is called the Primal problem, and the associated one is called the Dual problem. Either of the two can be treated as primal and the other as its dual. The two problems, thus, constitute a primal-dual pair.

Theorems

Theorem 1: The dual of the dual is primal.

Theorem2:

If Maximize Z =CX, Subject to A x ≤ B

Then dual of this primal is

Minimize $Z = B^T Y$

Subject to $A^T Y \geq C^T$

Where Y is the variable associated with dual problem

$$A= \begin{pmatrix} a11 & a12....ain \\ a21 & a22....a2n \\ am1 & .... amn \end{pmatrix} \quad C = \begin{pmatrix} c1 & c2..............cn \end{pmatrix} \quad X = \begin{pmatrix} x1 & x2..............xn \end{pmatrix} \quad B= \begin{pmatrix} b1 & b2............bm \end{pmatrix}$$

(a) If minimize Z = CX

Subject to AX ≥ B,

The dual of this primal is,

Maximize $Z = = B^T Y$

Subject to $A^T Y \geq C^T$

Where Y is the variable associated with the dual problem

$$Y= \begin{pmatrix} Y1 \\ Y2 \\ - \\ - \\ Yn \end{pmatrix}$$ associated with B

Example

Given the primal problem,

Maximize Z = $4 X_1 + 10 X_2 + 25 X_3$

Subject to constraints $2 X_1 + 4 X_2 + 8 X_3 \leq 25$

$4 X_1 + 9X_2 \; 8 X_3 \leq 30$

$6X_1 + 8X_2 + 2 X_3 \leq 40$

$X_{1,} X_{2,} X_3 \leq 0$

Formulate its dual problem.

Solution:

Here $C = [4 \quad 10 \quad 25]$

$X = [X1 \quad X2 \quad X3]$

$$A = \begin{pmatrix} 2 & 4 & 8 \\ 4 & 9 & 8 \\ 6 & 8 & 2 \end{pmatrix} \qquad B = \begin{pmatrix} 25 \\ 30 \\ 40 \end{pmatrix}$$

$B^T = [25 \quad 30 \quad 40]$

$$A^T = \begin{pmatrix} 2 & 4 & 8 \\ 4 & 9 & 8 \\ 6 & 8 & 2 \end{pmatrix} \qquad C^T = \begin{pmatrix} 4 \\ 10 \\ 25 \end{pmatrix}$$

$$Y^T = \begin{pmatrix} Y1 \\ Y2 \\ y3 \end{pmatrix}$$

$$\text{Min } Z = [25 \quad 30 \quad 40] * \begin{pmatrix} Y1 \\ Y2 \\ Y3 \end{pmatrix}$$

→ Min Z = 25 y1 + 30Y2 +40Y3

Subject to

$$\begin{pmatrix} 2 & 4 & 6 \\ 4 & 9 & 8 \\ 8 & 8 & 2 \end{pmatrix} \begin{pmatrix} Y1 \\ Y2 \\ Y3 \end{pmatrix} \geq \begin{pmatrix} 4 \\ 10 \\ 25 \end{pmatrix}$$

Solving this,

$2y1 + 4y2 + 6y3 \geq 4$

$4y1 + 9y2 + 8y3 \geq 10$

$8y1 + 8y2 + 2y3 \geq 25$

Is the dual of the given primal problem.

<u>Exercise for students: Find the dual of this dual problem, and show that the dual of the dual is the primal (original set of equations).</u>

ASSIGNMENT PROBLEM

- Is a special type of Transportation Problem in which each source should have the capacity to fulfill the demands of any of the destinations.
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Operator

$$
\begin{array}{c|cccccccc|}
 & 1 & 2 & \ldots & j & \ldots & m \\
\hline
1 & t_{11} & t_{12} & \ldots & t_{1j} & \ldots & t_{jm} \\
2 & t_{21} & t_{22} & \ldots & t_{2j} & \ldots & t_{2m} \\
3 & t_{31} & t_{32} & \ldots & t_{3j} & \ldots & t_{3m} \\
\vdots & \vdots & \vdots & & \vdots & & \vdots \\
i & t_{i1} & t_{i2} & \ldots & t_{ij} & \ldots & t_{im} \\
\vdots & \vdots & \vdots & & \vdots & & \vdots \\
m & t_{m1} & t_{m2} & \ldots & t_{mj} & \ldots & t_{mm} \\
\hline
\end{array}
$$

Where, m is the number of jobs as well as number of operators.

EXAMPLES:

ROW ENTRY	COLUMN ENTRY	CELL ENTRY
Jobs	Operator	Processing time
Operators	Machine	Processing time
Drivers of company vehicle	Routes	Travel time
physicians	Treatments	Number of cases handled

Model for assignment problem:

$$\text{Minimize } Z = \sum_{i=1}^{m}\sum_{j=1}^{m} C_{ij}X_{ij}$$

Subject to

$$\sum_{i=1}^{m} X_{ij} = 1 \qquad j = 1, 2, \ldots, m$$

$$\sum_{j=1}^{m} X_{ij} = 1 \qquad i = 1, 2, \ldots, m$$

where, X_{ij} = 0 or 1 for j = 1, 2, ... , m and i = 1, 2, ... , m
C_{ij} is processing time/cost etc.
Thus X_{ij} = 1 if i is assigned to column j otherwise X_{ij} = 0.

In this model, the objective function minimizes the total cost of assigning the rows to the columns.

The first set of constrains ensures that each column is assigned to only one row.

The second set of constrains ensures that row is assigned to only one column.

To solve the assignment problem we use Hungarian Method.

HUNGARIAN METHOD

It consists of two phases:

Phase 1: Row & Column reduction

Step 0: Consider te given cost matrix.

Step 1: obtain the next matrix by subtracting the minimum value of each row from the entries of that row.

Step 2: obtain the next matrix by subtracting the minimum value of each column from the entries of that column.

Now treat this matrix as the input for phase 2.

Phase 2: Optimization of the problem

Step 3: Draw a minimum number of lines to cover all the zeros of the matrix, The procedure for doing this involves the following steps:

3.1 Row scanning

I. Start from the first row, ask the following question: Is there exactly one zero in the row? If yes mark a square around that zero entry and draw a vertical line passing through that zero; otherwise skip that row.

II. After scanning the last row, check whether all the zeros are covered with lines. If yes, go to step 4; otherwise go to step 3.2

3.2 Column scanning

I. Starting from first column, ask the following question: Is there exactly one zero in that column? If yes mark a square around that zero entry and draw a horizontal line passing through that zero; otherwise skip that column.

II. After scanning the last column, check whether all the zeros are covered with lines. If yes, go to step 4; otherwise go to step 3.1.

Step 4: Check whether the number of squares marked is equal to the number of rows of the matrix. If yes, go to step 7; otherwise go to step 5.

Step 5: Identify the minimum value of the undeleted cell values. Obtain the next matrix by following these steps:

5.1 copy the entries on the lines but not on the intersection points of the present matrix as such without any modification to the corresponding positions of the next matrix.

5.2 copy the entries at the intersection points of the present matrix after adding the minimum undeleted cell value to the corresponding positions of the next matrix.

5.3 subtract the minimum undeleted cell value from all the undeleted cell values and then copy them to the corresponding positions of the next matrix.

Step 6: Go to step 3.

Step 7: treat the solution as marked by the squares as the optimal solution.

Example:

Lets solve the following problem using Hungarian method

operators

Jobs		1	2	3	4	5	row min
	1	10	12	15	12	8	8
	2	7	16	14	14	11	7
Jobs	3	13	14	7	9	9	7
	4	12	10	11	13	10	10
	5	8	13	15	11	15	8

After row reduction we get the following matrix,

2	4	7	4	0
0	9	7	7	3
6	7	0	2	2
2	0	1	3	0
0	5	7	3	7

Column min : 0 0 0 2 0

After column reduction, and assigning zeros according to step 3 we get the following matrix,

2	4	7	2	[0]
[0]	9	7	5	3
6	7	[0]	0	2
2	[0]	1	1	0
0	5	7	1	7

Here we observe that the number of marked zeros is not equal to number of rows, so go for next iteration by applying step 5,

2	4	6	1	[0]
[0]	9	7	4	4
7	8	[0]	0	3
2	[0]	0	0	0
0	5	6	[0]	7

Now we observe that number of marked zeros is equal to number of rows, hence optimal solution is given as follows:

Job	Operator	Time
1	5	8
2	1	7
2	3	7
4	2	10
5	4	11

Total time = 43 hours