Transportation problem

It is a special kind of LPP in which goods are transported from a set of sources to a set of destinations subjects to the supply and demand of the source and destination, respectively, such that the total cost of transportation is minimized.

Let m→ no. of sources

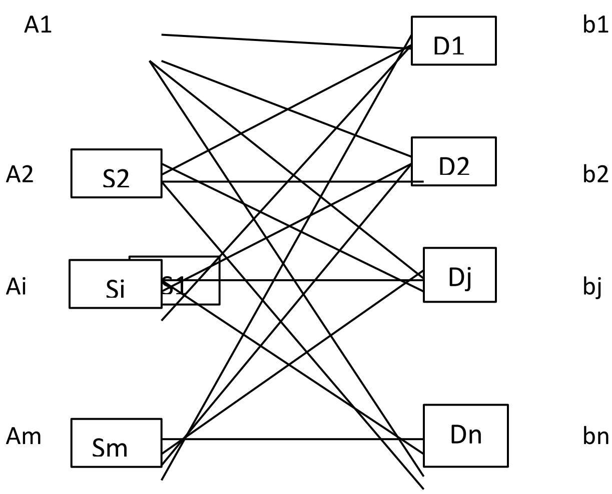
n→ no. of destinations

a1→ supply at the source i.

bj → Demand at the destination J

Cij → Cost of transportation/ unit from source I to destination j.

Xij → no. of units to be transported from source I to destination j



Schematic diagram of Simple TP

Mathematical Model for TP

Minimize Z $= \sum_{i=1}^{m} \sum_{j-1}^{n} \text{Cij Xij}$

Subject to $\sum_{j=1}^{n} \text{Xij} \leq \text{ai} \quad i = 1,2, \ldots \ldots m$

And

$\sum_{i=1}^{m} \text{Xij} \geq \text{bj} \quad j = 1,2, \ldots \ldots n$

Xij ≥ 0

Destination

Source 1	C11	C12--------CIJ---------C1n
2	C21	C22-------C2J----------C2n
i	Ci1	Ci2--------Cij-----------Cin
m	Cm1	Cm2--------cmj---------Cmn
Demand	b1	b2 --------- bj ---------- bn

example: Consider the following TP. Develop a LP model for this problem

		Destination 1	2	3	Supply
	1	20	10	15	200
source	2	10	12	9	300
	3	25	30	18	500
Demand		200	400	400	1000

Let Xij be the no. of units to be transported from the source I to the destination j, where I =1, 2, 3

J = 1 , 2, 3

Therefore a model of this problem →

Minimize Z = $\sum_{i=1}^{m} \sum_{j-1}^{n} \text{Cij Xij}$ { m=3, m=3}

Minimize Z = C11 X11 + C12 X12 + C13 X13
+C21 X21 + C22 X22 + C23 X23
+C31 X31 + C32 X32 + C33 X33

→ Minimize Z = 20x11 + 10x12 +15x13
+ 10x21 + 12 x22 + 9x23
+25 X31 + 30 x32 + 18 x33

Subject to $\sum_{j=0}^{n} \text{Xij} \leq \text{ai}$ $\quad i = 1,2,3$

→ X11 +X12 + X13 ≤ 200
X21 + X22 + X23 ≤ 300
X31 + X32 +X33 ≤ 500

$$\sum_{j=0}^{n} \text{Xij} \geq \text{bj} \qquad j = 1,2,3$$

X11 + X21 + X31 ≥ 200

X12 + X22 + X32 ≥ 400

X13 + X23 + X33 ≥ 400

Xij ≥ 0 where I = 1, 2, 3
J= 1, 2, 3

Types of Transportation Problem

1 Balanced TP

2 Unbalanced TP

Balanced Transportation Problem

If sum of the supplies of all the source is equal to the sum of the demands of all the destination then it is a balanced TP

i.e $\sum_{1=1}^{m} \mathrm{ai} = \sum_{j=1}^{n} \mathrm{bj}$

Unbalanced TP

If sum of the supplies of all the source is not equal to the sum of the demands of all the destination then it is an unbalanced TP, i.e., $\sum_{1=1}^{m} \mathrm{ai} \neq \sum_{j=1}^{n} \mathrm{bj}$

There are various methods for solving Transportation Problem. The solution procedure for transportation problem consists of two phases:

I. Phase 1: Finding the initial basic feasible solution.
II. Phase 2: optimization of the initial basic feasible solution which is obtained in Phase 1.

Here we are going to discuss Vogel's Approximation Method, which is one of the most efficient methods, and hence important.

Vogel's approximation method (VAM)

Algorithm

1. Find row penalties, i.e. the difference between the first minimum and the second minimum in each row. If the two minimum values are equal, then the row penalty is zero.
2. Find column penalties, i.e. the difference between the first minimum and the second minimum in each column. If the two minimum values are equal, then the column penalty is zero.
3. Find the maximum amongst the row penalties and column penalties and identify whether it occurs in a row or a column (break tie randomly). If the maximum penalty is in a row go to step 4, else go to step 7.
4. Identify the cell for allocation which has the least cost in that row.
5. Find the minimum of the supply and demand values with respect to the selected cell.
6. Allocate this minimum value to that cell and subtract this minimum value from the supply and demand values with respect to the selected cell and go to step 10.
7. Identify the cell for allocation which has the least cost in that column.
8. Find the minimum of the supply and demand values with respect to the selected cell
9. Allocate this minimum value to that cell and subtract this minimum value from the supply and demand values with respect to the selected cell.
10. Check whether exactly one of the rows and columns corresponding to the selected cell has zero supply/zero demand, respectively. If yes go to step 11, else go to step 12.
11. Delete the row/column which has zero supply/zero demand and revise the corresponding row/column penalties. Go to step 13.
12. Delete both the row and column with respect to the selected c ell. The revise the row and column penalties.
13. Check whether exactly one row/column is left out. If yes go to step 14, else go to step 3.
14. Match the supply/demand of the left-out row/column with the remaining demands/supplies of the undeleted columns/rows.

Example:

	1	2	3	4	supply	Penalty
1	3	1	7	4	300	(1-3=2)
2	2	6	5	9	400	(2-5)= 3*
3	8	3	3	2	500	(3-2)=1
Demand	250	350	400	200		
Penalty	1	2	2	2		

Find max of these penalties from both rows & columns (**First check if the demand = supply, for a balanced TP and then proceed)**

	1	2	3	4	
1	3	1	7	4	300
2	[250] 2	6	5	9	~~400~~ 150
3	8	3	3	2	500
	~~250~~ 0	350	400	200	

Row penalty

[300] 1	7	4	300	3 *
6	5	9	150	1
3	3	2	500	1
350	400	200		

Col penalty 2 2 2

	2	3	4		penalty
2	6	5	9	150	1
3	3	3	[200] 2	~~500~~ 300	0
	50	400	~~200~~ 0		
Penalty	3	2	7		

Here as penalty 7 corresponds to column 4 , allocation will be done in least cost cell of column 4

	~~2~~	3		Penalty
2	6	5	150	1
3	~~3~~ [50]	3	~~300~~ 250	0
	~~50~~	400		
Penalty	3	2		

	3		
2	5 [150]	~~150~~	5 *
3	3 [250]	250	3
	~~400~~		
	2		

Total cost = 2850

Degeneracy in LP

- Some iterations will be carried out without any improvement in the objective function.
- Infinite optimal solutions
- Redundant constraints

MODI METHOD (U-V method) [MODI fied DIstribution]

- This method is used to resolve degeneracy
- It tests for optimal solution

If no. of basic cells = m+n-1,
Where m→ no. of rows
n→ no. of occurs.
Then degeneracy occurs.

Algorithm

1. Row 1, row 2, .., row m of the cost matrix are assigned with variables u1, u2,…, um respectively and the column 1, column 2, …, column n are assigned with variables v1, v2,…, vn respectively.
2. Check whether the number of basic cells in the set of initial basic feasible is equal to m + n – 1. If yes go to step 4, else go to step 3.
3. Convert the necessary number of non-basic cells into basic cells to satisfy the condition stated in step 2 (while doing this, it should be seen that no closed loop formation is there with the inclusion of the new basic cell(s)). The concept of closed loop is explained in step 8.
4. Compute the values for u1, u2,…, um and v1, v2,…, vn by applying the following formula to all the basic cells only: $u_i + v_j = c_{ij}$ (assume u1 = 0)
5. Compute penalties/degeneracy D_{ij} for the non-basic cells by using the formula:

$$P_{ij} = u_i + v_j - c_{ij}$$

6. Check whether all D_{ij} values are less than or equal to zero. If yes go to step 12, else go to step 7.
7. Identify the non-basic cell which has the maximum positive penalty, and term that cell as the new basic cell.
8. Starting from the new cell, draw a closed loop consisting of only horizontal and vertical lines passing through some basic cells. Change of direction of the loop should be with 90 degrees only at some basic cell.
9. Starting from the new basic cell, alternatively assign + and – signs at the corners of the closed loop.
10. Find the minimum of the allocations made amongst the negatively signed cells.
11. Obtain the table for the next iteration by doing the following steps and then go to step 2:
    a. Add the minimum allocation obtained in the previous step to all the positively signed cells and subtract minimum allocation from all the negatively signed cells and then treat the net allocations as the allocations in the corresponding cells of the next iteration.
    b. Copy the allocations which are on the closed loop but not at the corner points of the closed loop, as well as the allocations which are not on the loop as such without any modifications to the corresponding cells of the next iteration.
12. The optimality is reached. Treat the present allocations to the set of basic cells as the optimum allocations.
13. Stop.

Example: Consider the following transportation problem involving three sources and four destinations. The cell entries represent the cost of transportation per unit.

a) Find the initial basic feasible solution
b) Optimize the solution by using U-V method

	1	2	3	4	
1	- 250 3	+ 50 1	7	4	300
2	+ 2	- 300 6	100 5	9	400
3	8	3	300 3	200 2	500
	250	350	400	200	

a) Total cost = 3x250 + 1x50 + 6x300 + 5x100 + 3x300 + 2x200 = 4400 using VAM

b) Here m+n-1= no. of allocated cell (4+3-1 = 6)

U1=0	V1=3	P12=1
U2= 5	V2= 1	P14 = -5
U3= 3	V3= 0	P21= 6
	V4= -1	P24= -5
		P31= -2
		P32 = 1

All Pij should be= < 0 for optimal solution.

As it is not so, we select the most positive P value and make a closed loop starting from that new basic cell, in this case cell(2,1) corresponding to P21, and go for the next iteration.

Min { (250-ϕ), (300-ϕ)} = 0
→ 250-ϕ=0 → ϕ=250

Cij = Ui + Vj → for basic cells (occupied)
Pij = Ui + Vj – Cij → for non basic cell (unoccupied)

	1	2	3	4
1	3	300 1	7	4
2	250 2	50 6	100 5	9
3	8	3	300 3	200 2

M+n-1= no. of allocated cell

U1=0	V1=-3	P11=-3-3= 0
U2= -5	V2= 1	P13 = 0-7= -7
U3= 3	V3= 0	P14= -1-4= -5
	V4= -1	P24= -6-9= -1
		P31= 0-8=-8
		P32 = 4-3=1

Here again all Pij is not equal to or less than zero. There is degeneracy at P32 carrying a positive value. So we assign a new basic cell at cell(3,2) corresponding to P32, and start making a closed loop starting from this cell.

Min {(50-ϕ), (300-ϕ)} =0
→ 50-ϕ=0 → ϕ=50

	1	2	3	4
1	3	300 1	7	4
2	250 2	6	150 5	9
3	8	50 3	250 3	200 2

M+n-1= no. of allocated cell

U1=0	V1=2	P11=-2-3= -5
U2= 4	V2= 1	P14= 0-4= -4
U3= 2	V3= -1	P22= 4+1-6=-1
	V4= 0	P31 = 2-2-8= -8

Now all the Pij < 0 hence optimal solution.
Total cost = 1x300 + 2x250+5x150+3x50+3x250+2x200 = 2850, which is a much improved solution as compared to that obtained in part (a)

Sequencing Problem

1. N jobs 2 machines.
2. N jobs 3 machines.
3. N jobs k machines.

n jobs 2 machines.

JOBS	J1	J2	J3	J4	J5	J6
Processing Time On M1 (Printing Machine)	1	3	8	5	6	3
Processing Time On M2 (Binding Machine)	5	6	3	2	2	10

M1 → ← M2

J1	J6	J2	J3	J4	J5

JOBS	Machine M1		Machine M2		IDLE TIME OF M2
	IN	OUT	IN	OUT	
J1	0	1	1	6	1
J6	1	4	6	16	-
J2	4	7	16	22	-
J3	7	15	22	25	-
J4	15	20	25	27	-
J5	20	26	27	29	-

n jobs 3 machines.

JOBS	A	B	C	D	E	F	G
M1	3	8	7	4	9	8	7
M2	4	3	2	5	1	4	3
M3	6	7	5	11	5	6	12

Solution:

The problem can be converted into n jobs 2 machines if either or both of the following conditions are satisfied.

1. Min (M1) >= Max (M2)
2. Min (M3) >= Max (M2)

So here condition "2" is satisfied.

Let G_i and H_i be 2 fictitious machines, then

$G_i = M_{1i} + M_{2i}$ $\quad$ i = 1, 2, 3, ... n

$H_i = M_{2i} + M_{3i}$ $\quad$ i = 1, 2, 3, ... n

JOBS	A	B	C	D	E *	F *	G *
G_i	7 *	11	9	9	10	12	10
H_i	10	10	7 *	16	6	10	15

$G_i \rightarrow$ $\leftarrow H_i$

A	D	G	B	F	C	E

JOBS	Machine M1		Machine M2		Machine M3	
	IN	OUT	IN	OUT	IN	OUT
A	0	3	3	7	7	13
D	3	7	7	12	13	24
G	7	14	14	17	24	36
B	14	22	22	25	36	43
F	22	30	30	34	43	49
C	30	37	37	39	49	54
E	37	43	43	44	54	59

Answer : 59 (total time)