## ASSIGNMENT PROBLEM:

Assignment problem is a special LPP which deals with assignment of workers to machines, clerks to various checkout counters, salesman to different sales areas, service crews to different districts and so on. Assignment is a problem because people possess varying abilities for performing different jobs and therefore, the costs of performing those jobs by different people are different. Obviously, if all persons could do a job in the same time or at the same cost then it would not matter who among them is assigned the job. Thus in an assignment problem, the question is how the assignments should be made in order that the total cost involved is minimized ( or the total value is maximized when pay offs are given in terms of, say, profits).

There are four methods of solving an assignment problem namely

- Complete enumeration method
- Transportation method
- Simplex method
- Hungarian assignment method

The first three methods are not efficient in handling assignment problems.

## HUNGARIAN ASSIGNMENT METHOD (HAM):

This method is specially designed to handle assignment problems in an efficient way. HAM is based on the concept of opportunity cost. For a typical balanced assignment problem involving a certain number of people and an equal number of jobs, with an objective function of minimization type, the method is applied as listed in the following steps.

**Step 1:** Locate the smallest cost element in each row of the cost table and subtract it from each element in that row. As a result there shall be at least one zero in each row of this new table, called the reduced cost table.

**Step 2:** In the reduced cost table obtained, consider each column and locate the smallest element in it. Subtract the smallest value from every other entry in the column. As a result of this , there would be at least one zero in each of the rows and columns of the second reduced cost table.

**Step 3:** Draw the minimum number of horizontal and vertical lines (not diagonal lines) that are required to cover all 0 elements. If the number of lines drawn is equal to n (i.e. the number of rows or columns) the solution is optimal and proceeds to step 6. If the number of lines drawn is less than n, go to step 4.

**Step 4:** Select the smallest uncovered cost element. Subtract this element from all uncovered elements including itself and add this element to each value located at the intersection of any two lines. The cost elements through which only one line passes remain unaltered.

**Step 5:** Repeat step 3 and 4 until an optimal solution is obtained.

**Step 6:** Given the optimal solution, make the job assignments as indicated by the zero elements. This can be done as follows:

a) Locate a row which contains only one 'zero' element. Assign the job corresponding to this element to its corresponding person. Cross out the zeros, if any, in the column corresponding to the element, which is indicative of the fact that the particular job and person are no more available.
b) Repeat (a) for each of such rows, which contain only one zero. Similarly, perform the same operation in respect of each column containing only one zero element, crossing out the zero(s), if any, in the row in which the element lies.
c) If there is no row or column with only a single 0 element left, then select a row or column arbitrarily and choose one of the jobs (or persons) and make the assignment. Now cross the remaining zeros in the column and row in respect of which the assignment is made.
d) Repeat steps (a) through (c) until all assignments are made.
***e)*** Determine the total cost with reference to the original cost table.

***Example:*** A production supervisor is considering how he should assign the four jobs that are to be performed, to four of the workers. He wants to assign the jobs to the workers such that the aggregate time to perform the jobs is the least. Based on previous experience, he has the information on the time taken by the four workers in performing these jobs, as given in the table:

| **Worker** | **Job** | | | |
|---|---|---|---|---|
| | A | B | C | D |
| **1** | 45 | 40 | 51 | 67 |
| **2** | 57 | 42 | 63 | 55 |
| **3** | 49 | 52 | 48 | 64 |
| **4** | 41 | 45 | 60 | 55 |

**Table 1: Time taken (in minutes) by 4 Workers**

**Solution:** The solution to this problem has been discussed in a step wise manner:

Step 1: The minimum elements of each row is subtracted from all elements in the row as shown in the following table known as the reduced cost table or opportunity cost table:

| **Worker** | **Job** | | | |
|---|---|---|---|---|
| | A | B | C | D |
| **1** | 5 | 0 | 11 | 27 |
| **2** | 15 | 0 | 21 | 13 |
| **3** | 1 | 4 | 0 | 16 |
| **4** | 0 | 4 | 19 | 14 |

Step 2: For each column of the above table the minimum value is subtracted from all other values giving us another reduced cost table:

| **Worker** | **Job** | | | |
|---|---|---|---|---|
| | A | B | C | D |
| **1** | 5 | 0 | 11 | 14 |
| **2** | 15 | 0 | 21 | 0 |
| **3** | 1 | 4 | 0 | 3 |
| **4** | 0 | 4 | 19 | 1 |

Step 3: Draw the minimum number of lines covering all zeros. We will first cover those rows/columns which contain larger number of zeros such that we obtain the following reduced cost table.

| **Worker** | **Job** | | | |
|---|---|---|---|---|
| | A | B | C | D |
| **1** | 5 | 0 | 11 | 14 |
| **2** | 15 | 0 | 21 | 0 |
| **3** | 1 | 4 | 0 | 3 |
| **4** | 0 | 4 | 19 | 1 |

Step 4: Since the number of lines drawn=4(=n), the optimal solution is obtained. The assignments are made after scanning the rows and columns for unit zeros. Assignments made are shown with squares as depicted in the table.

| **Worker** | **Job** | | | |
|---|---|---|---|---|
| | A | B | C | D |
| **1** | 5 | 0 | 11 | 14 |
| **2** | 15 | 0 | 21 | 0 |
| **3** | 1 | 4 | 0 | 3 |
| **4** | 0 | 4 | 19 | 1 |

Assignments are made in the following order. Since row 1, 3 and 4 contain 1 zero each, we assign 1-B, 3-C and 4-A. Since worker 1 has been assigned the job B, we cross the zero in the second column of the second row. After making these assignments, only worker 2 and job D are left for assignment. The final pattern of assignments is 1-B, 2-D, 3-C and 4-A involving a total time of 40+55+48+41=184 minutes ( as taken from the original table).

**UNBALANCED ASSIGNMENT PROBLEMS:**

The Hungarian method of solving assignment problems require that the number of columns should be equal to the number of rows in which case the problem is known as balanced problem and when the rows and columns are unequal, it is called an unbalanced assignment problem. In case of such class of problems, one to one match is not possible.

In such situations, dummy columns / rows, whichever is smaller in number are inserted with zeros as the cost elements. For example, in case of 4x5 cost matrix, a dummy row is added. In each column in respect of this row, a zero would be placed. After this process of adding dummy rows/ columns, the problem is solved in a usual manner.

***Example:*** You are given information about cost of performing different jobs by different persons. The job-person marking x indicates that the individual involved cannot perform the particular job. Using this information, state (a) the optimal assignment of jobs (b) the cost of such assignment.

| **Person** | **Job** | | | | |
|---|---|---|---|---|---|
| | $J_1$ | $J_2$ | $J_3$ | $J_4$ | $J_5$ |
| **$P_1$** | 27 | 18 | X | 20 | 21 |
| **$P_2$** | 31 | 24 | 21 | 12 | 17 |
| **$P_3$** | 20 | 17 | 20 | X | 16 |
| **$P_4$** | 22 | 28 | 20 | 16 | 27 |

Solution: Balancing the problem and assigning a high cost to the pairings $P_1$-$J_3$ and $P_3$-$J_4$, we obtain the following cost table:

| **Person** | **Job** | | | | |
|---|---|---|---|---|---|
| | $J_1$ | $J_2$ | $J_3$ | $J_4$ | $J_5$ |
| **$P_1$** | 27 | 18 | M | 20 | 21 |
| **$P_2$** | 31 | 24 | 21 | 12 | 17 |
| **$P_3$** | 20 | 17 | 20 | M | 16 |
| **$P_4$** | 22 | 28 | 20 | 16 | 27 |
| **$P_5$(dummy)** | 0 | 0 | 0 | 0 | 0 |

From the above table a reduced cost table is derived as depicted below. However the cells with prohibited assignments continue to be shown with the cost element M, as M is defined to be extremely large such that subtraction or addition of a value does not practically affect it.To test optimality, lines are drawn to cover all zeros.

| **Person** | **Job** | | | | |
|---|---|---|---|---|---|
| | $J_1$ | $J_2$ | $J_3$ | $J_4$ | $J_5$ |
| **$P_1$** | 9 | 0 | M | 2 | 3 |
| **$P_2$** | 19 | 12 | 9 | 0 | 5 |
| **$P_3$** | 4 | 1 | 4 | M | 0 |
| **$P_4$** | 6 | 12 | 4 | 0 | 11 |
| **$P_5$(dummy)** | 0 | 0 | 0 | 0 | 0 |

Since the number of lines covering zeros is less than n, we select the lowest uncovered cell which equals 4. With this value we obtain the revised reduced cost table as depicted below:

| Person | Job | | | | |
|---|---|---|---|---|---|
| | $J_1$ | $J_2$ | $J_3$ | $J_4$ | $J_5$ |
| **$P_1$** | 9 | 0 | M | 6 | 3 |
| **$P_2$** | 15 | 8 | 5 | 0 | 1 |
| **$P_3$** | 4 | 1 | 4 | M | 0 |
| **$P_4$** | 2 | 8 | 0 | 0 | 7 |
| **$P_5$(dummy)** | 0 | 0 | 0 | 4 | 0 |

Since in the above table, the number of lines covering zeros equals 5(=n), an optimal assignment can be made which is : $P_1$-$J_2$, $P_2$-$J_4$, $P_3$-$J_5$, $P_4$-$J_3$ while job $J_1$ would remain unassigned. The assignment pattern would cost 18+12+16+20=66 in aggregate.

**Maximization Case:**

In some situations the assignment problem may call for maximization of profit, revenue etc. as the objective. For dealing with such problems, we first change it into an equivalent minimization problem. This is achieved by subtracting each of the elements of the given pay-off matrix from a constant value (say K). Usually the largest of all values in the given matrix is located and then each one of the values is subtracted from it. Then the problem is solved the same way as the minimization problem.

***Example:*** A company plans to assign 5 salesmen to 5 districts in which it operates. Estimates of sales revenue in thousands of rupees for each salesman in different districts are given in the following table. In your opinion, what should be the placement of the salesmen if the objective is to maximize the expected sales revenue?

**Table 2: Expected Sales data**

| Salesman | District | | | | |
|---|---|---|---|---|---|
| | $D_1$ | $D_2$ | $D_3$ | $D_4$ | $D_5$ |
| **$S_1$** | 40 | 46 | 48 | 36 | 48 |
| **$S_2$** | 48 | 32 | 36 | 29 | 44 |
| **$S_3$** | 49 | 35 | 41 | 38 | 45 |
| **$S_4$** | 30 | 46 | 49 | 44 | 44 |
| **$S_5$** | 37 | 41 | 48 | 43 | 47 |

Solution:

Since this is a maximization problem, we first subtract each of the entries in the table from the largest one (i.e. 49) to obtain the following opportunity loss matrix:

| Salesman | District | | | | |
|---|---|---|---|---|---|
| | $D_1$ | $D_2$ | $D_3$ | $D_4$ | $D_5$ |
| **$S_1$** | 9 | 3 | 1 | 13 | 1 |
| **$S_2$** | 1 | 17 | 13 | 20 | 5 |
| **$S_3$** | 0 | 14 | 8 | 11 | 4 |
| **$S_4$** | 19 | 3 | 0 | 5 | 5 |
| **$S_5$** | 12 | 8 | 1 | 6 | 2 |

Now we can proceed the same way as in case of minimization problems by following the below mentioned steps:

Step 1: Subtract minimum value in each row from every value in the row so as to obtain the following reduced cost table:

| Salesman | District | | | | |
|---|---|---|---|---|---|
| | $D_1$ | $D_2$ | $D_3$ | $D_4$ | $D_5$ |
| **$S_1$** | 8 | 2 | 0 | 12 | 0 |
| **$S_2$** | 0 | 16 | 12 | 19 | 4 |
| **$S_3$** | 0 | 14 | 8 | 11 | 4 |
| **$S_4$** | 19 | 3 | 0 | 5 | 5 |
| **$S_5$** | 11 | 7 | 0 | 5 | 1 |

Step 2:Now subtract minimum value in each column from each value in that column in the above reduced cost table to obtain the following table. Test for optimality by drawing lines to cover zeros.

| Salesman | District | | | | |
|---|---|---|---|---|---|
| | $D_1$ | $D_2$ | $D_3$ | $D_4$ | $D_5$ |
| **$S_1$** | 8 | 0 | 0 | 7 | 0 |
| **$S_2$** | 0 | 14 | 12 | 14 | 4 |
| **$S_3$** | 0 | 12 | 8 | 6 | 4 |
| **$S_4$** | 19 | 1 | 0 | 0 | 5 |
| **$S_5$** | 11 | 5 | 0 | 0 | 1 |

Since the number of lines covering all zeros is fewer than n, we select the least uncovered cell value, which equals 4 and obtain a modified table as shown below:

| Salesman | District | | | | |
|---|---|---|---|---|---|
| | $D_1$ | $D_2$ | $D_3$ | $D_4$ | $D_5$ |
| **$S_1$** | 12 | 0 | 0 | 7 | 0 |
| **$S_2$** | 0 | 10 | 8 | 10 | 0 |
| **$S_3$** | 0 | 8 | 4 | 2 | 0 |
| **$S_4$** | 23 | 1 | 0 | 0 | 5 |
| **$S_5$** | 15 | 5 | 0 | 0 | 1 |

There are more than one optimal assignments possible in this case because of existence of multiple zeros in different rows and columns. The following assignments are possible:

$S_1$-$D_2$,$S_2$-$D_1$,$S_3$-$D_5$,$S_4$-$D_3$,$S_5$-$D_4$ or

$S_1$-$D_2$,$S_2$-$D_5$,$S_3$-$D_1$,$S_4$-$D_3$,$S_5$-$D_4$ or

$S_1$-$D_2$,$S_2$-$D_5$,$S_3$-$D_1$,$S_4$-$D_4$,$S_5$-$D_3$ or

$S_1$-$D_2$,$S_2$-$D_1$,$S_3$-$D_5$,$S_4$-$D_4$,$S_5$-$D_3$

Each of these assignment patters would lead to an expected aggregate sales equal to 231 thousand rupees.

**PRACTICE QUESTIONS**

1. **To simulate interest and provide an atmosphere for intellectual discussion, the finance faculty in a management school decides to hold special seminars on four contemporary topics – leasing, portfolio management, private mutual funds, swaps and opinions. Such seminars would be held once per week in the afternoons. However, scheduling these seminars (one for each topic, and not more than one seminar per afternoon) has to be done carefully so that the number of students unable to attend is kept to a minimum. A careful study indicates that the number of students who cannot attend a particular seminar on a specific day is as follows:**

| | **Leasing** | **Portfolio Management** | **Private Mutual Funds** | **Swaps and Opinions** |
|---|---|---|---|---|
| **Monday** | 50 | 40 | 60 | 20 |
| **Tuesday** | 40 | 30 | 40 | 30 |
| **Wednesday** | 60 | 20 | 30 | 20 |
| **Thursday** | 30 | 30 | 20 | 30 |
| **Friday** | 10 | 20 | 10 | 30 |

**Find an optimal schedule of the seminars. Also find out the total number of students who will be missing at least one seminar.**

2. **A solicitor's firm employs typists on hourly piece-rate basis for their daily work. There are five typists and their charges and speeds are different. According to an earlier understanding, only one job is given to one typist and the typist is paid for a full hour even when he works for a fraction of an hour. Find the least cost allocation for the following data:**

| | **Rate/hour(Rs)** | **Number of pages typed/hour** | **Job** | **No.of Pages** |
|---|---|---|---|---|
| **A** | 5 | 12 | P | 199 |
| **B** | 6 | 14 | Q | 175 |
| **C** | 3 | 8 | R | 145 |
| **D** | 4 | 10 | S | 298 |
| **E** | 4 | 11 | T | 178 |

3. **A company has four sales representatives who are to be assigned to four different sales territories. The monthly sales increase estimated for each sales representative for different sales territories (in lacs of Rs) are shown in the following table:**

| **Sales Representatives** | **Sales Territories** | | | |
|---|---|---|---|---|
| | I | II | III | IV |
| **A** | 200 | 150 | 170 | 220 |
| **B** | 160 | 120 | 150 | 140 |
| **C** | 190 | 195 | 190 | 200 |
| **D** | 180 | 175 | 160 | 190 |

**Suggest optimal assignment and total maximum sales increase per month.**

**If for certain reasons, sales representative B cannot be assigned to sales territory III. Will the optimal assignment schedule be different? If so, find that schedule and the effect on total sales.**