DUALITY IN LPP:

Every LPP exists in pairs so that corresponding to every given LPP, there is another LPP which is related to it and therefore, may be derived from it. Before writing a dual to the given primal LPP it is necessary to express it in standard form if it is not so, such that:

- All the variables in the problem should be non negative
- All constraints should be of $\leq$ type if the problem is of maximization type and $\geq$ type if the problem is of minimization type.

The necessary changes are made as follows:

When all variables in the LPP are not non-negative:

If a variable in an LPP is unrestricted in sign so that is a free variable, then it may be replaced by the difference of two non-negative variables, each of which is non-negative.

When all constraints are not in the right direction:

If a constraint involves an inequality in a direction opposite to the one desired, then it is multiplied by -1 throughout and the direction of the inequality is reversed. E.g. If a constraint is given as $7x_1 - 5x_2 \geq 4$ and the desired inequality direction is $\leq$, then this constraint is replaced by $-7x_1 + 5x_2 \leq -4$.

For constraints involving equations, we obtain inequalities by replacing it with a pair of inequalities in the opposite direction. E.g. $7x_1 + 5x_2 = 42$ is replaced by a pair of following inequalities:

$$7x_1 + 5x_2 \leq 42 \quad and \quad 7x_1 + 5x_2 \geq 42$$

One of the above equations is multiplied by -1 to bring the inequality in the desired direction.

GENERAL METHOD FOR WRITING A DUAL OF LPP:

Consider the primal problem of the following form:

$$\begin{aligned} &Maximize && Z = cx \\ &subject\ to && ax \leq b \\ & && x \geq 0 \end{aligned}$$

where c= row matrix containing the coefficients in the objective function

x=column matrix containing the decision variables

a=matrix containing the coefficients in the constraints b=column matrix containing the R.H.S values of the constraints.

The example of the above representation is:

$$\begin{aligned} &Maximize && Z = 40x_1 + 35x_2 \\ &subject\ to && 2x_1 + 3x_2 \le 60 \\ &&& 4x_1 + 3x_2 \le 96 \\ &&& x_1, x_2 \ge 0 \end{aligned}$$

$$c = \begin{bmatrix} 40 & 35 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad a = \begin{bmatrix} 2 & 3 \\ 4 & 3 \end{bmatrix}, \quad b = \begin{bmatrix} 60 \\ 96 \end{bmatrix}$$

The dual corresponding to the above generalized problem is as shown:

$$\begin{aligned} &Minimize && G = b'y \\ &subject\ to && a'y \ge c' \\ &&& y \ge 0 \end{aligned}$$

where b'= transpose of the b matrix of the primal problem

a'=transpose of the coefficients matrix of the primal problem

c'=transpose of the matrix of the objective function coefficients of the primal problem

y=column matrix of the dual variables

Example: Write the dual for the LPP given below.

$$\begin{aligned} &Maximize && Z = 40x_1 + 35x_2 \\ &subject\ to && 2x_1 + 3x_2 \le 60 \\ &&& 4x_1 + 3x_2 \le 96 \\ &&& x_1, x_2 \ge 0 \end{aligned}$$

Solution: The above problem can be reproduced in the general form as:

$$Maximize \quad Z = \overset{c}{\begin{bmatrix} 40 & 35 \end{bmatrix}} \overset{x}{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}$$

subject to

$$\overset{a}{\begin{bmatrix} 2 & 3 \\ 4 & 3 \end{bmatrix}} \overset{x}{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}} \le \overset{b}{\begin{bmatrix} 60 \\ 96 \end{bmatrix}}$$

Obtaining the values for dual,

$$b' = \begin{bmatrix} 60 & 96 \end{bmatrix}, \; a' = \begin{bmatrix} 2 & 4 \\ 3 & 3 \end{bmatrix}, \; c' = \begin{bmatrix} 40 \\ 35 \end{bmatrix}, \; y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

The dual for the above problem in terms of matrix form is as shown below:

$$\text{Minimize} \quad G = \overset{b'}{\begin{bmatrix} 60 & 96 \end{bmatrix}} \overset{y}{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}}$$

subject to

$$\overset{a'}{\begin{bmatrix} 2 & 4 \\ 3 & 3 \end{bmatrix}} \overset{y}{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}} \geq \overset{c'}{\begin{bmatrix} 40 \\ 35 \end{bmatrix}}$$

The above obtained dual can be rewritten as:

$$\begin{aligned} &\text{Minimize} && G = 60y_1 + 96y_2 \\ &\text{subject to} && 2y_1 + 4y_2 \geq 40 \\ &&& 3y_1 + 3y_2 \geq 35 \\ &&& y_1, y_2 \geq 0 \end{aligned}$$

Example: Write the dual for the LPP given below.

$$\begin{aligned} &\text{Maximize} && Z = 8x_1 + 10x_2 + 5x_3 \\ &\text{subject to} && x_1 - x_3 \leq 4 \\ &&& 2x_1 + 4x_2 \leq 12 \\ &&& x_1 + x_2 + x_3 \geq 2 \\ &&& 3x_1 + 2x_2 - x_3 = 8 \\ &&& x_1, x_2, x_3 \geq 0 \end{aligned}$$

Solution: Firstly the above problem needs to be expressed in standard form.

Constraints 1 and 2: Since both these are of the type $\leq$, they need not be modified.

Constraint 3: Since this constraint is of the type $\geq$, it needs to be converted to $\leq$ type, by multiplying both sides with -1 such that the constraint becomes $-x_1 - x_2 - x_3 \leq -2$.

Constraint 4: Since this constraint is in the form of an equation, it can be represented by a pair of inequalities as shown:

$$\begin{aligned} 3x_1 + 2x_2 - x_3 \leq 8 \\ 3x_1 + 2x_2 - x_3 \geq 8 \end{aligned}$$

The second inequality can be converted from $\geq$ type to $\leq$ type by multiplying -1 on both sides such that the equation becomes $-3x_1 - 2x_2 + x_3 \leq -8$.

Thus the given problem can be re represented in the standard form as follows:

$$\begin{aligned} &\textit{Maximize} \quad & Z &= 8x_1 + 10x_2 + 5x_3 \\ &\textit{subject to} \quad & & x_1 - x_3 \le 4 \\ & & & 2x_1 + 4x_2 \le 12 \\ & & & -x_1 - x_2 - x_3 \le -2 \\ & & & 3x_1 + 2x_2 - x_3 \le 8 \\ & & & -3x_1 - 2x_2 + x_3 \le -8 \\ & & & x_1, x_2, x_3 \ge 0 \end{aligned}$$

The matrix form of the above primal is as shown:

$$\textit{Maximize} \quad Z = \overset{c}{\begin{bmatrix} 8 & 10 & 5 \end{bmatrix}} \overset{x}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}$$

subject to

$$\overset{a}{\begin{bmatrix} 1 & 0 & -1 \\ 2 & 4 & 0 \\ -1 & -1 & -1 \\ 3 & 2 & -1 \\ -3 & -2 & 1 \end{bmatrix}} \overset{x}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}} \le \overset{b}{\begin{bmatrix} 4 \\ 12 \\ -2 \\ 8 \\ -8 \end{bmatrix}}$$

Next we obtain the values to formulate the dual of the above problem.

$$b' = \begin{bmatrix} 4 & 12 & -2 & 8 & -8 \end{bmatrix}, \ a' = \begin{bmatrix} 1 & 2 & -1 & 3 & -3 \\ 0 & 4 & -1 & 2 & -2 \\ -1 & 0 & -1 & -1 & 1 \end{bmatrix}, c' = \begin{bmatrix} 8 \\ 10 \\ 5 \end{bmatrix}, \ y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

The dual for the above problem in terms of matrix form is as shown below:

$$\textit{Minimize} \quad G = \overset{b'}{\begin{bmatrix} 4 & 12 & -2 & 8 & -8 \end{bmatrix}} \overset{y}{\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix}}$$

subject to

$$\overset{a'}{\begin{bmatrix} 1 & 2 & -1 & 3 & -3 \\ 0 & 4 & -1 & 2 & -2 \\ -1 & 0 & -1 & -1 & 1 \end{bmatrix}} \overset{y}{\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix}} \ge \overset{c'}{\begin{bmatrix} 8 \\ 10 \\ 5 \end{bmatrix}}$$

The above obtained dual can be rewritten as:

$$\begin{aligned} &\textit{Minimize} && G = 4y_1 + 12y_2 - 2y_3 + 8y_4 - 8y_5 \\ &\textit{subject to} && y_1 + 2y_2 - y_3 + 3y_4 - 3y_5 \geq 8 \\ & && 4y_2 - y_3 + 2y_4 - 2y_5 \geq 10 \\ & && -y_1 - y_3 - y_4 + y_5 \geq 5 \\ & && y_1, y_2, y_3, y_4, y_5 \geq 0 \end{aligned}$$

We know that corresponding to an n-variable, m-constraint primal problem, there should be m-variable, n-constraint dual problem. However in the above given example the primal problem involves 3 variable and 4 constraints which means the dual should have 4 variables and 3 constraints. But the dual we have obtained contains 5 variable. In order to resolve this inconsistency, we express $(y_4 - y_5) = y_6$, a variable unrestricted in sign. Hence the dual can be rewritten as:

$$\begin{aligned} &\textit{Minimize} && G = 4y_1 + 12y_2 - 2y_3 + 8y_6 \\ &\textit{subject to} && y_1 + 2y_2 - y_3 + 3y_6 \geq 8 \\ & && 4y_2 - y_3 + 2y_6 \geq 10 \\ & && -y_1 - y_3 - y_6 \geq 5 \\ & && y_1, y_2, y_3 \geq 0, y_6 \textit{ unrestricted in sign} \end{aligned}$$

Thus whenever a constraint in the primal involves an equality sign, its corresponding dual variable shall be unrestricted in sign. Similarly, an unrestricted variable in the primal would imply that the corresponding constraint shall bear an =sign.

COMPARING THE OPTIMAL SOLUTIONS OF PRIMAL AND DUAL

Example: Consider the following primal and its corresponding dual:

Primal

$$\begin{aligned} &\textit{Maximize} && Z = 40x_1 + 35x_2 \\ &\textit{subject to} && 2x_1 + 3x_2 \leq 60 \\ & && 4x_1 + 3x_2 \leq 96 \\ & && x_1, x_2 \geq 0 \end{aligned}$$

Dual

$$\begin{aligned} &\textit{Minimize} && G = 60y_1 + 96y_2 \\ &\textit{subject to} && 2y_1 + 4y_2 \geq 40 \\ & && 3y_1 + 3y_2 \geq 35 \\ & && y_1, y_2 \geq 0 \end{aligned}$$

Upon solving both the primal and its dual using simplex method, the optimal solution table so obtained is depicted below:

Optimal solution for the primal

Basis		x_1	x_2	S_1	S_2	b_i
x_2	35	0	1	2/3	-1/3	8
x_1	40	1	0	-1/2	1/2	18
c_j		40	35	0	0	
Solution		18	8	0	0	
$\Delta_j = c_j - z_j$		0	0	-10/3	-25/3	

Optimal solution for the dual

Basis		y_1	y_2	S_1	S_2	A_1	A_2	b_i
y_2	96	0	1	-1/2	1/3	1/2	-1/3	25/3
y_1	60	1	0	1/2	-2/3	-1/2	2/3	10/3
c_j		60	96	0	0	M	M	
Solution		10/3	25/3	0	0	0	0	
$\Delta_j = c_j - z_j$		0	0	18	8	M-18	M-8	

On comparing both the primal and dual optimal solutions , the following observations were made:

- The objective function values of both the problems are same . Thus with $x_1 = 18$ and $x_2 = 8$ the value of Z comes out to be 1000. Similarly for the dual of the problem, with $y_1 = 10/3$ and $y_2 = 25/3$, the value of also comes out to be 1000.
- The numerical value of each of the variables in the optimal solution to the primal is equal to the value of its corresponding variable in the dual contained in Δ_j row. Thus in the primal problem $x_1 = 18$ and $x_2 = 8$ whereas in the dual $S_1 = 18$ and $S_2 = 8$ (in the Δ_j row).
- Similarly the numerical value of each of the variables in the optimal solution to the dual is equal to the value of its corresponding variable in the primal (in the Δ_j row). Thus, $y_1 = 10/3$ and $y_2 = 25/3$ in the dual and $S_1 = 10/3$ and $S_2 = 25/3$ in the primal (only absolute values are considered).

Note: If feasible solution exists for both the primal and dual problems then both problems have optimal solutions of which objective function values are equal.
A peripheral relationship between them is that if one problem has an unbounded solution, its dual has no feasible solution.