GAME THEORY

Game theory is a body of knowledge which is concerned with the study of decision making in situations where two or more rational opponents are involved under conditions of competition and conflicting interests. It deals with human processes in which an individual decision making unit who can be an individual, a group, a formal or informal organization, or a society, is not in complete control of the other decision making units, the opponent(s), and is addressed to problems involving conflict, co-operation or both at various levels.

The main objective in the theory of games is to determine the rules of rational behavior in the game situations, in which the outcomes are dependent on the actions of the interdependent players. A game refers to a situation in which two or more players are competing. It involves the players who have different goals or objectives and whose fates are intertwined. They are in a situation in which there may be a number of possible outcomes with different values to them. Although they might have some control that would influence the outcome, they do not have complete control over others. Labour unions striking against the company management is an example of a game.

In a game situation, each of the players has a set of strategies available. A strategy refers to the action to be taken by a player in various contingencies in playing a game. There is a set of outcomes each of which is the result of the particular choices of strategies made by the players on a given play of the game, and pay-offs are accorded to each player in each of the possible outcomes. Each of the players being assumed to be rational, his preference ordering of the different outcomes is determined by the order of magnitudes of the associated pay-offs, and since, in general, the orders of magnitudes of the pay-offs accruing to the players in different outcomes do not coincide, a game models a situation in which there are conflicts of interests.

The players in the game strive for optimal strategies. An optimal strategy is such as provides the best situation in the game in the sense that it involves maximal pay-off to the players.

GAME MODELS:

Game models can be classified on the basis of number of factors as discussed below:

Number of players involved:

- *Two person game:* In a game situation, when interests of two persons are in conflict
- *N-person game:* When interest of more than two persons are in conflict. In this game model, it does not necessarily imply that exactly n people would be involved, but rather that the participants can be classified into n mutually exclusive categories and members of each of the categories have identical interests.

Sum of gains or losses:

- *Zero-Sum Game:* In a game, if the sum of gains or losses is equal to zero it is called zero-sum or constant-sum game.
- *Non-zero sum game:* If the sum of gains or losses is not equal to zero, it is called a non-zero sum game.

Number of strategies employed in the game:

- *Finite:* A game is said to be finite if each player has the option of choosing from only a finite number of strategies.
- *Infinite:* A game is said to be infinite if each player does not have the option of choosing from a finite number of strategies.

TWO-PERSON ZERO SUM GAMES:

This type of game involves two persons such that the gain made by one equals the loss incurred by other. Suppose that there are two firms A and Bin an area, which for a long period in the past, have been selling a competing product and are now engaged in struggle for a larger share of the market. Now with the total market of a given size, any share of the market gained by one firm must be lost by the other, and therefore the sum of gains and losses equals 0.

Assume both these firms are considering the same three strategies in a bid to gain the share in the market; low advertising, high advertising and quality improvement. We assume that currently they are sharing the market equally and further each of the firms can employ only one strategy at a time.

Under these conditions, there are a total of 3x3=9 combinations of the moves possible – thus, low advertising by firm A may be accompanied by low advertising, high advertising or quality improvement by firm B and so on for other strategies. The strategies of low advertising, high advertising and quality improvement have been marked as a1,a2 and a3 respectively for firm A and b1,b2 and b3 respectively for firm B.

		B's Strategy		
		b1	b2	b3
A's Strategy	**a1**	12	-8	-2
	a2	6	7	3
	a3	-10	-6	2

This pay-off matrix is drawn from A's point of view – a positive pay-off indicates that firm A has gained the market share at the expense of firm B while a negative pay-off matrix implies B's gain at A's expense.

The problem is to determine the best strategy for A and B, assuming that they both are acquainted with the information contained in the pay-off matrix and that each one is not aware of the move the other is likely to take. The conservative approach to the selection of best strategy would call for a cautious attitude of assuming the worst and acting accordingly. In reference to the given pay-off matrix, if firm A employs strategy a1, it would expect firm B to employ b2, thereby reducing A's pay-off from strategy a1 to its minimum value(=-8), representing a loss to firm A. Similarly if firm A employs a2, it would assume firm B to employ strategy b3 that would give A a 3% increase in the market share and corresponding to a3 strategy by firm A , it would expect firm B to use b1,which puts firm A at a loss of 10 points.

Firm A would like to make the best use of the situation by choosing the maximum of these minimal pay-offs i.e. the one which would ensure it the highest of the minimum rewards associated with these. This decision rule of selection is called maximin strategy. Thus A will select a2 as its strategy.

Similarly, firm B would also employ a cautious approach in choosing its strategy. Thus, when firm B adopts b1, it expects A to employ a1, which would ensure it (firm A) the maximum advantage. Similarly for b2 and b3 firm B's expectation would be associated with employment of a2. To minimize the advantage accruing to A, firm B would select a strategy that would yield least advantage to its competitor- strategy b3. This decision of B is in accordance with minimax strategy.

Corresponding to maximin rule of firm A and minimax rule of firm B, the pay-off is 3. This amount is the worth or value of the game and represents the final pay-off to the winning player by the losing player. Since the pay-off matrix is drawn from A's point of view, if the value of the game is positive, it is favourable to A otherwise it is favourable to B. The game is said to be equitable if its value equals zero, in which case it favours none.

SADDLE POINT:

When the maximin value equals minimax value, the game is solved and it refers to the point of equilibrium. This point of equilibrium is known as the saddle point.

Operationally, to obtain the saddle point, if it exists, we determine the minimum pay-off value for each row and also the maximum pay off value for each of the columns. In case the largest of the row minima is equal to the smallest of the column maxima, then it represents a saddle point as shown in the example below:

		B's Strategy			
		b1	b2	b3	Row Minima
A's Strategy	a1	12	-8	-2	-8
	a2	6	7	3	3*
	a3	-10	-6	2	-10
Column Maxima		12	7	3*	

In the above tale 3 represents the saddle point.

In some problems there may be two saddle points as depicted in the following example:

		B's Strategy				
		b1	b2	b3	b4	Row Minima
A's Strategy	**a1**	4	-16	14	-15	-16
	a2	-6	7	-4	-6	-6*
	a3	6	-2	0	-6	-6*
Column Maxima		6	7	14	-6*	

This game has two saddle points represented by a2b4 and a3b4. The value of the game is -6 indicating a net loss of 6 points to A and equivalent profit to B.

WHEN NO SADDLE POINT EXISTS:

All games don't have a saddle point in which case it is not possible to find its solution in terms of pure strategies- the maximin and minimax. Games without saddle points are not strictly determined. The solution to such problems calls for employing mixed strategies. A mixed strategy represents a combination of two or more strategies that are selected one at a time, according to predetermined probabilities. Thus, in employing a mixed strategy, a player decides to mix his choices among several alternatives in a certain ratio.

Example: The following is the pay-off matrix of a game being played by A and B. Determine the optimal strategies for the players and the value of the game.

		B's Strategy	
		b1	b2
A's Strategy	**a1**	8	-7
	a2	-6	4

Solution: Since no saddle point exists in this case an alternative method is neede to determine the strategies followed by A and B respectively.

Suppose A plays strategy a1 with probability x and strategy a2 with probability 1-x. If B plays strategy b1, then A's expected pay-off can be determined in reference to the figures given in column 1 of the pay-off matrix as follows:

Expected pay-off(given that B plays b1)= 8x-6(1-x)

Expected pay-off(given that B plays b2)= -7x+4(1-x)

Now we can determine the value of x, so that the expected pay-off for A is the same irrespective of the strategy adopted by B

Thus, 8x-6(1-x)=7x+4(1-x)

Or 8x-6+6x=-7x+4-4x

Or x=10/25=2/5

A would do best to adopt strategies a1 and a2, choosing in a random manner, in the proportion 2:3(i.e. 2/5 and 3/5). The expected pay-off for A using this mixed strategy is:

8x(2/5)-6(1-2/5) = 8x(2/5)-6(3/5)= -2/5

Or -7(2/5)+4(1-2/5) = -7(2/5)+4(3/5) = -2/5

Thus A shall have a net loss of 2/5 per play in the long run.

Similarly, mixed strategy for B can be determined by supposing B plays strategy b1 with probability y and strategy b2 with probability 1-y. Thus,

Expected pay-off(given that A plays a1)= 8y-7(1-y)

Expected pay-off(given that A plays a2)= -6y+4(1-y)

We can determine the value of y, which will ensure equal pay-off irrespective of the strategy adopted by A

Thus, 8y-7(1-y)=-6y+4(1-y)

Or 8y-7+7y=-6y+4-4y

Or y=11/25

Thus, B should play strategies b1 and b2 in the ratio of 11:14 in a random manner

The expected pay-off for B using this mixed strategy is:

8x(11/25)-7(14/25) = -10/25 =-2/5

Or -6(11/25)+4(14/25) = -10/25= -2/5

Thus B shall gain 2/5 per play in the long run.

We conclude that A and B should both use mixed strategies as given in the below table and the value of the game equals -2/5.

	Strategy	Probability
For A,	a1	2/5
	a2	3/5
For B,	b1	11/25
	b2	14/25

GENERAL FORMULA FOR A ZERO-SUM TWO PERSONS GAME:

Suppose A and B have strategies a1,a2 and b1,b2 respectively and the pay-offs as given below then, if x is the probability with which A chooses strategy a1 and y is the probability that B chooses strategy b1, we have

		B's Strategy	
		b1	b2
A's Strategy	**a1**	a_{11}	a_{12}
	a2	a_{21}	a_{22}

$$x = \frac{a_{22} - a_{21}}{(a_{11} + a_{22}) - (a_{12} + a_{21})} \;;\; y = \frac{a_{22} - a_{12}}{(a_{11} + a_{22}) - (a_{12} + a_{21})} \;;\; V = \frac{a_{11}a_{22} - a_{12}a_{21}}{(a_{11} + a_{22}) - (a_{12} + a_{21})}$$

We can cross check the above formula by finding the value of x and y for the above example using this formula, as shown below:

$$x = \frac{4-(-6)}{(8+4)-(-7-6)} = \frac{10}{25} = \frac{2}{5} \;;\; y = \frac{4-(-7)}{(8+4)-(-7-6)} = \frac{11}{25} \;;\; V = \frac{8\times 4-(-7)(-6)}{(8+4)-(-7-6)} = \frac{-10}{25} = \frac{-2}{5}$$

These values match with the ones already obtained.

DOMINANCE RULE:

In a game, sometimes a strategy available to a player might be found to be preferable to some other strategy/strategies. Such a strategy is said to *dominate* the other one(s). This concept of domination is very usefully employed in simplifying the games and thus helps in finding solutions to the games. Let us consider the pay-off matrix in respect of our first illustration of two firms *A* and *B*, which is reproduced hereafter.

		B's Strategy		
		b1	b2	b3
A's Strategy	**a1**	12	-8	-2
	a2	6	7	3
	a3	-10	-6	2

Consider the strategies open to firm A. We notice that every element of the second row exceeds the corresponding element of the third row. Clearly, A shall never prefer to play a3, because in comparison to this strategy, it shall be better off in adopting a2, regardless of what strategy is adopted by B. Thus, a3 is dominated by a2 and hence can be deleted.

		B's Strategy		
		b1	b2	b3
A's Strategy	**a1**	12	-8	-2
	a2	6	7	3

From the reduced matrix we observe that the first column values are larger than their counterparts in the third column. Since B would like to minimize the pay-offs to A, it would always prefer to choose b3 to b1. Thus, strategy b1 is dominated by strategy b3. Now, deleting b1, we have

		B's Strategy	
		b2	b3
A's Strategy	**a1**	-8	-2
	a2	7	3

The second row values in this matrix are larger than the corresponding values of the first one and, therefore, a2 dominates a1. Furthermore deletion of a1 yields only a2 open to firm A. Since firm B loses more by choosing b2, it would adopt b3. Thus the strategies are a2 and b3, and a value 3 as the solution to this reduced 1x1 game.

Thus, by using dominance rule on one players strategies we can get a reduced game. Then we can use domination on the strategies of the other player to further reduce the game, if possible. In this way, moving forward and backward we reduce the game as much as possible. If saddle point of the game exists, then naturally the game shall be reduced to a 1x1 game. When the saddle point does not exist then, if after application of the dominance rule, a game becomes of the order 2x2, then the mixed strategies can be determined by using the analytical method. When after the domination rule is applied, the game is reduced to 2xn or mx2 size, it can be handled graphically. In any event, whenever the game is reduced in size by applying the dominance rule then the solution of the reduced game is also the solution of the original game.

Example: Solve the following game:

		B's Strategy	
		b1	b2
A's Strategy	**a1**	28	0
	a2	2	12
	a3	4	7

Solution: In this matrix none of the elements in any row are smaller than the elements in the other two rows.

However we notice that A would never play strategy a3, because for B's b1 strategy, A would play a1 (which will earn him maximum profit) and for B's b2 strategy, A would play a2. Thus, rows a1 and a2 dominate row a3 and hence a3 can be eliminated. Now the problem is reduced to the following matrix:

		B's Strategy	
		b1	b2
A's Strategy	**a1**	28	0
	a2	2	12

Now applying the general formula, we obtain

$$x = \frac{a_{22} - a_{21}}{(a_{11} + a_{22}) - (a_{12} + a_{21})} = \frac{12 - 2}{(28 + 12) - (0 + 2)} = \frac{10}{38} = \frac{5}{19}$$

$$y = \frac{a_{22} - a_{12}}{(a_{11} + a_{22}) - (a_{12} + a_{21})} = \frac{12 - 0}{(28 + 12) - (0 + 2)} = \frac{12}{38} = \frac{6}{19}$$

$$V = \frac{a_{11}a_{22} - a_{12}a_{21}}{(a_{11} + a_{22}) - (a_{12} + a_{21})} = \frac{28 \times 12 - 0 \times 2}{(28 + 12) - (0 + 2)} = \frac{336}{38} = \frac{168}{19}$$

Thus the optimal strategy for A is(5/19,14/19,0) and for B it is(6/19,13/19) and the value of the game is 168/19.

SOLUTION OF 2XN GAMES:

When the player A has only two strategies to choose from, and player B has n strategies, the game is of the order 2xn whereas in case B has only two strategies to choose from and A has m strategies then the game is of the order of mx2.

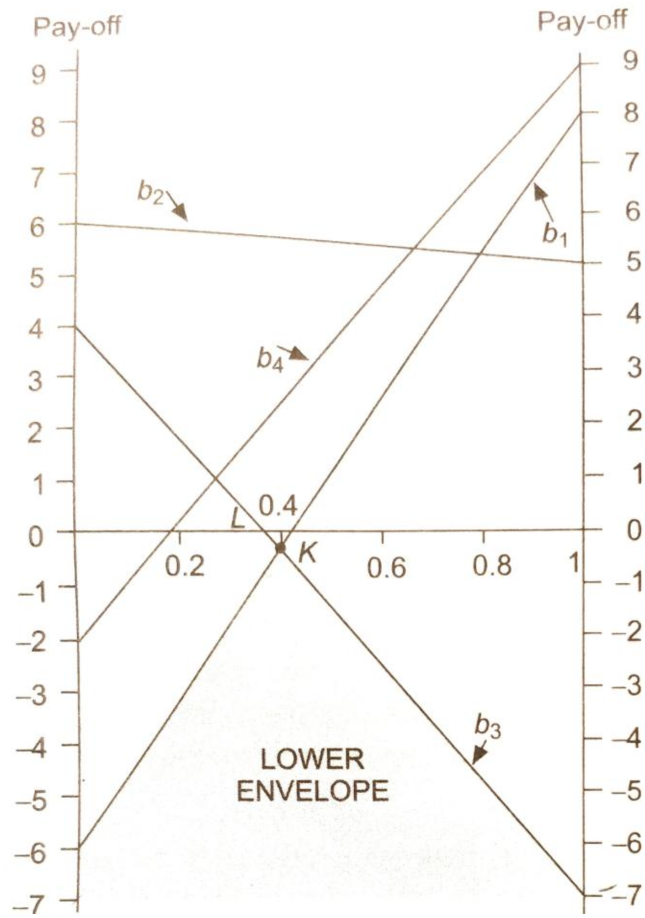
Such games can be solved by using the graphical method, which is aimed at reducing the game to the order of 2x2, by identifying and eliminating the dominated strategies, and then solve it by the analytical method used for solving such games. The resultant solution is also the solution to original problem.

***Example:* Solve the following game using graphical approach:**

		B's Strategy			
		b1	b2	b3	b4
A's Strategy	**a1**	8	5	-7	9
	a2	-6	6	4	-2

Solution: Suppose A plays strategies a1 and a2 with probabilities x and 1-x respectively. When B chooses to play b1, the expected payoff for A shall be 8x+(-6)(1-x) or 14x-6. Similarly, the expected payoff functions in respect of b2,b3 and b4 can be derived as being 6-x; 4-11x and 11x-2 respectively. All the expected pay-off functions can be represented graphically by plotting them as a function of x as shown in the below graph.

The lines are marked b1,b2, b3 and b4 and they represent the respective strategies. For each value of x, the height of the lines at that point denotes the pay-offs of each of B's strategies against (x,1-x) for A. A is concerned with his least pay-off when he plays a particular strategy, which is represented by the lowest of the four lines at that point, and wishes to choose x so as to maximize this minimum pay-off. This is represented by point K in the figure where the lower envelope (represented by the shaded region), the lowest of the lines at point, is the highest. This point lies at the intersection of the lines representing strategies b1 and b3. The distance KL=-0.4(-2/5) in the figure represents the game value, V, and x=OL (=0.4 or 2/5) is the optimal strategy for A.



Alternative method: Alternatively the game can be written as a 2x2 game as follows, with strategies a1 and a2 for A, and b1 and b3 for B.

		B's Strategy	
		b1	b3
A's Strategy	**a1**	8	-7
	a2	-6	4

Now applying the general formula, we obtain

$$x = \frac{a_{22} - a_{21}}{(a_{11} + a_{22}) - (a_{12} + a_{21})} = \frac{4 - (-6)}{(8 + 4) - (-7 - 6)} = \frac{10}{25} = \frac{2}{5}$$

$$y = \frac{a_{22} - a_{12}}{(a_{11} + a_{22}) - (a_{12} + a_{21})} = \frac{4 - (-7)}{(8 + 4) - (-7 - 6)} = \frac{11}{25}$$

$$V = \frac{a_{11}a_{22} - a_{12}a_{21}}{(a_{11} + a_{22}) - (a_{12} + a_{21})} = \frac{8 \times 4 - (-7)(-6)}{(8 + 4) - (-7 - 6)} = \frac{-2}{5}$$

Thus the optimal strategy for A is(2/5,3/5) and for B it is(11/25,0,14/25,0) .

PRACTICE QUESTIONS

1. **Shruti Ltd. Has developed a sales forecasting function for its products and the products of its competitors, Purnima Ltd. There are four strategies S1,S2,S3 and S4 available to Shruti Ltd. and three strategies to Purnima Ltd. The pay-offs corresponding to all the twelve combinations of the strategies are given below. From the table you can see that, for example, if strategy S1 is employed by Shruti Ltd. And strategy P1 by Purnima Ltd. then there shall be a gain of Rs. 30000 in quarterly sales to the former. Other entries can be similarly interpreted.**

		Purnima's Strategy		
		P1	P2	P3
Shruti's Strategy	**S1**	30,000	-21,000	1,000
	S2	18,000	14,000	12,000
	S3	-6,000	28,000	4,000
	S4	18,000	6,000	2,000

2. **In a small town, there are only two stores that handle sundry goods- ABC and XYZ. The total number of customers is equally divided between the two, because price and quality of the goods sold are equal. Both stores have good reputation in the community, and they render equally good customer service. Assume that a gain of customer by ABC is a loss to XYZ and vice versa. Both stores plan to run annual Diwali sales during the first week of November. Sales are advertised through a local**

newspaper, radio and television media. With the aid of an advertising firm, store ABC constructed the game matrix given below.
(Note: Figures in the matrix represent a gain or loss of customers.)

		XYZ's Strategy		
		Newspaper	Radio	Television
ABC's Strategy	**Newspaper**	30	40	-80
	Radio	0	15	-20
	Television	90	20	50

Determine optimal strategies and worth of such strategies for both ABC and XYZ.

3. **Reduce the following two-person-zero-sum game to 2x2 order and obtain the optimal strategies for each player and the value of the game:**

		Player B			
		B1	B2	B3	B4
Player A	**A1**	3	2	4	0
	A2	3	4	2	4
	A3	4	2	4	0
	A4	0	4	0	8

4. **A company is currently involved in negotiations with its union on the upcoming wage contract. With the aid of an outside mediator, the table below was constructed by the management group. The pulses are to be interpreted as proposed wage increases while a minus figure indicates that a wage reduction is proposed. The mediator informs the management group that he has been in touch with the union and that they have constructed a table that is comparable to the table developed by the management. Both the company and the union must decide on an overall strategy before negotiations begin. The management group understands the relationship of the company strategies to union strategies in the following table but lacks specific knowledge of game theory to select best strategy (or strategies) for the firm. Assist the management on this problem. What game value and strategies are available to the opposing groups.**

Conditional costs to company

		Union Strategies			
		U1	U2	U3	U4
Company Strategies	**C1**	+0.25	+0.27	+0.35	-0.02
	C2	+0.20	+0.16	+0.08	+0.08
	C3	+0.14	+0.12	+0.15	+0.13
	C4	+0.30	+0.14	+0.19	0.00

(in Lac Rs)