NUMERICAL AND STATISTICAL COMPUTING (MCA-202-CR)

Autumn Session

UNIT 3

DIFFERENTIAL EQUATION

A differential equation is an equation that in addition to independent and dependent variables also contains one or more derivatives of the dependent variables. These derivatives can be ordinary or partial.

As a solution of a differential equation,, instead of finishing with an expression, we end up with the numerical values taken by that expression. This is known as the numerical solution of the differential equation.

BASIC CONCEPTS AND TERMINOLOGY USED IN DIFFERENTIAL EQUATIONS:

1. ORDINARY DERIVATIVE: If y is a function of x, i.e. $y=f(x)$, then $\frac{dy}{dx}$ is called the ordinary derivative. It is the rate of change of the dependent variable with respect to the independent variable.
2. PARTIAL DERIVATIVE : If u is a function of x and y, i.e. u= f(x,y), then $\frac{\partial u}{\partial x}\Big|_y$ is called the partial derivative with respect to x keeping y constant, and $\frac{\partial u}{\partial x}\Big|_x$ is called the partial derivative with respect to y keeping x constant. It is the rate of change of the dependent variable with respect to one of the independent variable keeping others fixed.
3. ORDINARY DIFFRENTIAL EQUATION: It is an equation involving only ordinary derivatives of one or more functions with respect to single independent variables. Examples include:

   $$\frac{dy}{dx} = -\frac{x}{y}, \qquad \frac{dy}{dx} = x^2 + y^2, \qquad d^2y/dx^2 + y = 0$$

4. PARTIAL DIFFERENTIAL EQUATION: It is an equation involving only partial derivatives of one or more functions with respect to single independent variables. Examples include:

   $$\partial y / \partial t = k\frac{\partial^2 u}{\partial x^2}$$

5. ORDER OF A DIFFERENTIAL EQUATION: It is the order of the highest order derivative in the differential equation.
   For example:

m d^2y/dx^2 +c dy/dx +ky =0 has order=2,i.e., it's a second order ordinary differential equation.

6. DEGREE OF A DIFFERENTIAL EQUATION: It is the power of the highest order derivative in the differential equation.

For example: $(d^2y/dx^2)^2 + c\, dy/dx + ky = 0$ is a second degree ordinary differential equation.

ALGORITHMS TO SOLVE ORDINARY DIFFERENTIAL EQUATION:

1. EULER METHOD:

- It is a first order numerical procedure for solving ordinary differential equations for a given initial value.
- It is the oldest and simplest method but not as accurate or efficient as other methods.
- To improve its accuracy, the value of the step size 'h' should be as small as possible.
- In the initial value problem, the starting point of the solution curve and the slope of the curve at the starting point are given.

DERIVATION OF EULERS METHOD:

Consider a first order ordinary differential equation: $\frac{dy}{dx} = f(x, y)$ which can also be written as $y'(x) = f(x, y)$ with an initial condition $y = y_1$ for $x = x_1$.

i.e. $y'(x_1) = f(x_1, y_1)$.....................eq.1

Using the mean value theorem that states:

"If a function is continuous and differentiable between two points say (x_1, y_1) and (x_2, y_2), then the slope of the line joining these points is equal to the derivative of the function at least at one other point say (c, d) between these two points, i.e.

$y'(c) = (y(x_2) – y(x_1)) / (x_2-x_1)$..............eq. 2

If we substitute $c=x_1$ and $h=x_2-x_1$, then equation 2 can be written as :

$y'(x_1) = (y(x_2) – y(x_1)) / h$

$y(x_2) – y(x_1) = h\, \mathbf{y'(x_1)}$

Using eq.1 in above equation we have,

$y(x_2) – y(x_1) = hf(x_1,y_1)$

$y(x_2) = y(x_1) + hf(x_1,y_1)$

$y_2 = y_1 + hf(x_1,y_1)$.................. ......eq.3

Using eq3 we can find the second point on the solution curve as (x_2, y_2).

Similarly, taking (x_2, y_2) as the starting point, we have

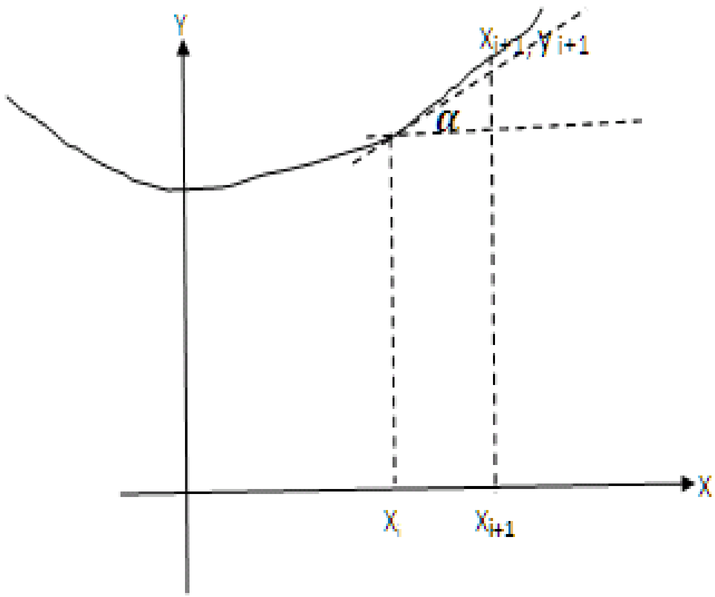
$$y_3 = y_2 + h\, f(x_2, y_2)$$

In general, the $(i+1)^{th}$ point on the solution curve is obtained from the i^{th} point using following formula:

$$\mathbf{y_{i+1} = y_i + h\, f(x_i, y_i)}$$

With each increasing *I* the value of x is added to step count.

GEOMETRIC DERIVATION:



Tan ∝ = Perpendicular / Base

$= (y_{i+1} - y_i) / (x_{i+1} - x_i)$

Also, Tan ∝$= f(x_i, y_i)$

Therefore,

$$f(x_i, y_i) = (y_{i+1} - y_i) / (x_{i+1} - x_i)$$
$$f(x_i, y_i) * (x_{i+1} - x_i) = (y_{i+1} - y_i)$$
$$f(x_i, y_i) * h = (y_{i+1} - y_i) \quad \text{where } h = x_{i+1} - x_i = \text{step size}$$
$$h\, f(x_i, y_i) + y_i = y_{i+1}$$

$\mathbf{y_{i+1} = y_i + h\, f(x_i, y_i)}$, which is the general equation for Euler's method.

Example

Q: Given dy/dx = xy with y(1) =5. Find solution correct to decimal positions in the interval [1,1.5] using step size h=0.1.

Sol: In our example, $f(x,y) = xy$

$X_1 = 1$, $Y_1 = 5$, $h = 0.1$

Modified Euler's formula = $y_{i+1} = y_i + hf(x_i,y_i)$

For i=1:

$y_2 = y_1 + h\, f(x_1,y_1)$

$= 5 + 0.1(1)(5)$

$= 5.5$

Therefore, $x_2 = 1+0.1 = 1.1$, $y_2 = 5.5$

For i=2:

$y_3 = y_2 + h\, f(x_2, y_2)$

$= 5.5 + 0.1(1.1)(5.5)$

$= 6.105$

Therefore, $x_3 = 1.1 + 0.1 = 1.2$, $y_3 = 6.105$

For i=3:

$y_4 = y_3 + h\, f(x_3, y_3)$

$= 6.105 + 0.1(1.2)(6.105)$

$= 6.838$

Therefore, $x_4 = 1.2 + 0.1 = 1.3$, $y_4 = 6.838$

For i=4:

$y_5 = y_4 + h\, f(x_4, y_4)$

$= 6.838 + 0.1(1.3)(6.838)$

$= 7.727$

Therefore, $x_5 = 1.3 + 0.1 = 1.4$, $y_5 = 7.727$

For i=5:

$y_6 = y_5 + h\, f(x_5, y_5)$

$= 7.727 + 0.1(1.4)(7.727)$

$= 8.809$

Therefore, $x_6 = 1.4 + 0.1 = 1.5$, $y_6 = 8.809$

I	1	2	3	4	5	6
X_i	1	1.1	1.2	1.3	1.4	1.5
Y_i	5	5.5	6.105	6.838	7.727	8.809

2. MODIFIED EULER METHOD:

It is one of the predictor –corrector formula, i.e. it is a multiple step method.

Single - Step Methods: In order to extrapolate the solution curve, they only use information available at previous point.eg. Euler's method.

Multiple - Step Method: In order to extrapolate the solution curve, they use past information of the curve, which should be 2 or more than 2 points. E.g. Modified Euler's method.

The problem with predictor – corrector method including Modified Euler's method is that they are not self starting i.e. values at first few points are computed using some other methods and then from there on these methods can take on.

Modified Euler's method works correctly only if the function is linear. The alternate method is to use the average slope within the interval. This is approximated by the mean of the slopes at both end points of the interval.

i.e. $(y'(x_i) + y'(x_{i+1}))/2$

then,

$$y_{i+1} = y_i + h/2\,[y'(x_i) + y'(x_{i+1})]$$

$$y_{i+1} = y_i + h/2\,[f(x_i,y_i) + f(x_{i+1},y_{i+1})]$$

this is an improved estimate for y_{i+1} at x_{i+1}.

But this method works by estimating or predicting the value of y_{i+1} by basic Euler's method.

Thus, the value of y_{i+1} is predicted using the equation

$$y^p_{i+1} = y_i + hf(x_i,y_i)$$

This is known as the "predictor formula". Using this predicted value of y, a more accurate value of y_{i+1} is computed using the following formula:

$$y^c_{i+1} = y_i + h/2\,[f(x_i,y_i) + f(x_{i+1}, y^p_{i+1})]$$

This equation is known as the "corrector formula".

Modified Euler's method is thus a two step method , comprising of following steps:

i. Predict y_{i+1} using Predictor formula.
ii. Correct y_{i+1} using Corrector formula.

Example

Q: Given dy/dx = xy with y(1) =5. Find solution correct to decimal positions in the interval [1,1.5] using step size h=0.1.

Sol: In our example, f(x,y) =xy

$X_1 = 1$, $Y_1 = 5$, $h = 0.1$

Modified Euler's formula = $y^p_{i+1} = y_i + hf(x_i, y_i)$ and $y^c_{i+1} = y_i + h/2\,[f(x_i, y_i) + f(x_{i+1}, y^p_{i+1})]$

For i=1:

$y^p_2 = y_1 + hf(x_1, y_1) \quad = 5 + 0.1(1)(5) \quad = 5.5$

$y^c_2 = y_1 + h/2\,[f(x_1, y_1) + f(x_2, y^p_2)]$

$= 5 + 0.1/2[f(1,5)+f(1.1,5.5)]$

$= 5 + 0.1/2\,[1*5 + 1.1*5.5]$

$= 5.553$

Therefore, $x_2 = 1.1$, $y_2 = 5.553$

For i=2:

$y^p_3 = y_2 + hf(x_2, y_2) \quad = 5.553 + 0.1(1.1)(5.553) \quad = 6.164$

$y^c_3 = y_2 + h/2\,[f(x_2, y_2) + f(x_3, y^p_3)]$

$= 5.553 + 0.1/2[f(1.1,5.553)+f(1.2, 6.164)]$

$= 5.553 + 0.1/2\,[1.1*5.553 + 1.2*6.164]$

$= 6.231$

Therefore, $x_3 = 1.2$, $y_3 = 6.231$

For i=3:

$y^p_4 = y_3 + hf(x_3, y_3) \quad = 6.231 + 0.1(1.2)(6.231) \quad = 6.979$

$y^c_4 = y_3 + h/2\,[f(x_3, y_3) + f(x_4, y^p_4)]$

$= 6.231+0.1/2[f(1.2, 6.231)+f(1.3,6.979)]$

$= 6.231+ 0.1/2\,[1.2*6.231+1.3*6.979]$

$= 7.059$

Therefore, $x_4 = 1.3$, $y_4 = 7.059$

For i=4:

$y^p_5 = y_4 + hf(x_4, y_4) \quad = 7.059 + 0.1(1.3)(7.059) \quad = 7.977$

$y^c_5 = y_4 + h/2\,[f(x_4, y_4) + f(x_5, y^p_5)]$

$= 7.059 + 0.1/2[f(1.3, 7.059)+f(1.4, 7.977)]$

$= 7.059 + 0.1/2 [1.3*7.059+1.4* 7.977]$

$= 8.8076$

Therefore, $x_5 = 1.4$, $y_5 = 8.076$

For i=5:

$y^p_6 = y_5 + hf(x_5,y_5) = 8.076+ 0.1(1.4)(8.076) = 9.207$

$y^c_6 = y_5 + h/2 [f(x_5,y_5) + f(x_6, y^p_6)]$

$= 8.076+ 0.1/2[f(1.4, 8.076)+f(1.5, 9.207)]$

$= 8.076 + 0.1/2 [1.4*8.076+1.5*9.207]$

$= 9.332$

Therefore, $x_6 = 1.5$, $y_6 = 9.332$

I	**1**	**2**	**3**	**4**	**5**	**6**
X_i	1	1.1	1.2	1.3	1.4	1.5
Y_i	5.0	5.553	6.231	7.059	8.076	9.332

3. RUNGE – KUTTA METHODS:

These methods were devised by C.Runge and extended by W. Kutta a few years later. The Runge – Kutta methods are actually a family of methods, of which the second order and fourth order methods are widely used. Much greater accuracy can be achieved by R.K.Methods. In these methods, first the slope at some of the intermediate points is computed and then the weighted average of slopes is used to extrapolate the next solution point.

RUNGE KUTTA FOURTH ORDER METHODS

In Runge – Kutta fourth order methods, the slope at four points including the starting point is computed, and then the weighted average of these slopes is computed as:

$S= 1/6 (S_1+2S_2+2S_3+S_4)$

Where

$S_1 = f (x_1,y_1)$

$S_2 = f (x_1+h/2, y_1+h/2S_1)$

$S_3 = f (x_1+h/2, y_1+h/2S_2)$

$S_4 = f (x_1+h, y_1+hS_3)$

The value of the dependent variable *'y'* is computed as:

$y_2 = y_1 + hS$

Example

Q: Given dy/dx = xy with y(1) =5. Find solution correct to decimal positions in the interval [1,1.3] using step size h=0.1.

Sol: The formula for fourth order runge – kutta methods is:

$y_2 = y_1 + hS$

Where

$S = 1/6\ (S_1 + 2S_2 + 2S_3 + S_4)$

$S_1 = f\ (x_1, y_1)$

$S_2 = f\ (x_1 + h/2,\ \ y_1 + h/2S_1)$

$S_3 = f\ (x_1 + h/2,\ \ y_1 + h/2S_2)$

$S_4 = f\ (x_1 + h,\ \ y_1 + hS_3)$

In our example, f(x,y) =xy

$X_1 = 1, \qquad Y_1 = 5, \qquad h = 0.1$

<u>For i=1:</u>

$y_2 = y_1 + 0.1 * S$

$S_1 = f\ (x_1, y_1) = f(1,5) = 1*5 = 5$

$S_2 = f\ (x_1 + 0.1/2,\ y_1 + 0.1/2 * S_1) = f(1.05, 5+0.05*5) = f(1.05, 5.25) = 1.05 * 5.25 = 5.513$

$S_3 = f\ (x_1 + 0.1/2,\ y_1 + 0.1/2 * S_2) = f(1.05, 5+0.05*5.513) = f(1.05, 5.276) = 1.05 * 5.276 = 5.540$

$S_4 = f\ (x_1 + 0.1,\ y_1 + 0.1 * S_3) = f(1.1, 5+0.1*5.540) = f(1.1, 5.554) = 1.1*5.554 = 6.109$

S=1/6[5+2(5.513) +2(5.540)+6.109] = 5.536

Therefore, $y_2 = 5 + 0.1*5.536 = 5.554$

Thus, $x_2 = 1.1, \qquad y_2 = 5.554$

<u>For i=2:</u>

$Y_3 = y_2 + 0.1 * S$

$S_1 = f\ (x_2, y_2) = f(1.1, 5.554) = 1.1*5.554 = 6.109$

$S_2 = f(x_2 + 0.1/2, y_2 + 0.1/2 * S_1) = f(1.15, 5.554+0.05*6.109) = f(1.15, 5.859) = 1.15*5.859 = 6.738$

$S_3 = f(x_2 + 0.1/2, y_2 + 0.1/2 * S_2) = f(1.15, 5.554+0.05*6.738) = f(1.15, 5.891) = 1.15*5.891 = 6.775$

$S_4 = f\ (x_2 + 0.1,\ \ y_2 + 0.1 * S_3) = f(1.2, 5.554+0.1*6.775) = f(1.2, 6.232) = 1.2*6.232 = 7.478$

$S=1/6[6.109+2(6.738) +2(6.775)+7.478] = 6.769$
Therefore, $y_3= 5.554+0.1*6.769 =6.231$
Thus, $x_3=1.2$, $y_3=6.231$

For i=3:

$Y_4 =y_3+0.1*S$

$S_1 = f (x_3,y_3) = f(1.2, 6.231) = 1.2* 6.231 = 7.477$

$S_2=f(x_3+0.1/2,y_3+0.1/2*S_1)=f(1.25,6.231+0.05*7.477)=f(1.25,6.604)=1.25*6.604 =8.256$

$S_3=f(x_3+0.1/2,y_3+0.1/2*S_2)=f(1.25,6.231+0.05*8.256)=f(1.25,6.644)=1.25*6.644= 8.305$

$S_4 = f (x_3+0.1, y_3+0.1 * S_3)= f(1.3,6.231+0.1*8.305)=f(1.3,7.062)=1.3*7.062 = 9.181$

$S=1/6[7.477+2(8.256) +2(8.305)+ 9.181] = 8.297$
Therefore, $y_4= 6.231+0.1*8.297 = 7.061$
Thus, $x_3=1.3$, $y_3=7.061$

I	**1**	**2**	**3**	**4**
X_i	1	1.1	1.2	1.3
Y_i	5.0	5.554	6.231	7.061

4. TRAPEZOIDAL RULE:

The trapezoidal rule approximates the area under the curve by connecting successive points on the curve to form trapezoids of the uniform width, and then summing the area under these trapezoids to obtain the approximate area under the curve.

DERIVATION:

For the derivation of the formula for the trapezoid rule, we assume a function f(x) given in following form:

X	X_1	X_1+h	$X_1+ 2h$	**…**	X_1+nh
Y=f(x)	Y_1	Y_2	Y_3	**…**	Y_{n+1}

We consider the trapezoid formed by connecting the points (x_k,y_k) and (x_{k+1},y_{k+1}). Further, we consider the first two terms of the Newton's forward difference interpolating polynomial to represent the straight line function f(x)

i.e. $y= f(x) =y_k +\Delta y_k u$

Where

$\Delta y_k = y_{k+1}- y_k$

$u = (x-x_k) / h$

$h = x_{k+1} -x_k$

Also, the area of k^{th} strip is given by

$$I_k = \int_{x_k}^{x_{k+1}} y dx \text{--------------eq1}$$

Since $u = (x-x_k) / h$ and $d(uv) = udv + vdu$

Therefore,

$d(hu) = udh + hdu$ $= d(hu) = udh + h [d((x-x_k) / h)]$

Also h and x_k are constant, so their differentiation is zero

$0+hdu=dx-0$

$hdu=dx$

$dx= hdu$------------------eq2

Substituting value of eq2 in eq 1 we get

$$I_k = \int_{x_k}^{x_{k+1}} y\, hdu$$

Since $y= y_k + \Delta y_k u$

Also at, $u=0$ at $x=x_k$ and $u=1$ at $x= x_{k+1}$

$$I_k = \int_0^1 [y_k + \Delta y_k u]\, hdu$$

$$I_k = h[\int_0^1 y_k\, du + \Delta y_k u\, du\,]$$

$$I_k = h\, [y_k \int_0^1 du + \Delta y_k \int_0^1 udu]$$

Since, $\int_a^b dx = x| \begin{matrix} b \\ a \end{matrix}$

Therefore, $\int_a^b xdx = (x^2/2)| \begin{matrix} b \\ a \end{matrix}$

$$I_k = h\, \left[y_k\, u \Big|_0^1 + \Delta y_k \frac{u^2}{2}\Big| \begin{matrix} 1 \\ 0 \end{matrix} \right]$$

$$I_k = h\, \left[y_k(1-0) + \Delta y_k \left(\frac{1^2}{2} - \frac{0^2}{2}\right) \right]$$

$$I_k = h\, [y_k + \Delta y_k /2\,]$$

$$I_k = h/2\, [2y_k + \Delta y_k\,]$$

Substituting, $\Delta y_k = y_{k+1} - y_k$

$$I_k = \frac{h}{2} [2y_k + (y_{k+1} - y_k\,]$$

$$\boldsymbol{I_k = \frac{h}{2} [y_k + y_{k+1}\,]}$$

This is the area under the curve between the points $x= x_k$ and $x=x_{k+1}$ i.e. the k^{th} strip. If the function is divided into *n* strips each of width *h*, the required integral over a range x=a and x=b is given by:

$I = I_1+I_2+I_3+\ldots+I_n$

$$I = \frac{h}{2}\left[y_1 + y_2\right] + \frac{h}{2}\left[y_2 + y_3\right] + \frac{h}{2}\left[y_3 + y_4\right] + \cdots + \frac{h}{2}\left[y_n + y_{n+1}\right]$$

$$I = \frac{h}{2}\left[y_1 + 2y_2 + 2y_3 + \cdots + 2y_n + y_{n+1}\right]$$

This is the formula for trapezoidal rule. It gives the correct value of the integral only if f(x) is a linear function.

Example

Q: Evaluate $\int_1^2 e^{-\frac{1}{2}x}\, dx$ using four intervals.

Sol: With four intervals, the interval size h = $\frac{2-1}{4}$ = 0.25, the function is calculated as

When x=1, y=f(x) = $e^{-\frac{1}{2}(1)} = 0.607$

When x=1.25, y=f(x) = $e^{-\frac{1}{2}(1.25)} = 0.535$

When x=1.50, y=f(x) = $e^{-\frac{1}{2}(1.50)} = 0.472$

When x=1.75, y=f(x) = $e^{-\frac{1}{2}(1.75)} = 0.417$

When x=2.0, y=f(x) = $e^{-\frac{1}{2}(2.0)} = 0.368$

Therefore, we have the following distribution:

X	1	1.25	1.50	1.75	2.0
f(x)	0.607	0.535	0.472	0.417	0.368

Substituting these values in trapezoidal rule formula

$$I = \frac{h}{2}\left[y_1 + 2y_2 + 2y_3 + \cdots + 2y_n + y_{n+1}\right]$$

we get,

$$I = \frac{0.25}{2}\left[0.607 + 2(0.535) + 2(0.472) + 2(0.417) + 0.368\right]$$

$$I = 0.478$$

5.1 SIMPSONS 1/3 RULE:

Simpson's $\frac{1}{3^{rd}}$ rule gives more accurate approximation of the integral value since it corrects 3 points on the curve by second order parabolas and then sums the area under parabolas to obtain the approximate area under the curve.

DERIVATION:

For the derivation of the formula of Simpson's $\frac{1}{3^{rd}}$ rule, we assume a function f(x) given in following form:

X	X_1	X_1+h	X_1+ 2h	…	X_1+nh
Y=f(x)	Y_1	Y_2	Y_3	…	Y_{n+1}

We consider that the parabola passes through the points (x_k,y_k) , (x_{k+1},y_{k+1}) and (x_{k+2},y_{k+2}). Further, we consider the first two terms of the Newton's forward difference interpolating polynomial to represent the straight line function f(x)

i.e. $y= f(x) =y_k +\Delta y_k u + (\Delta^2 y_k)/ 2! * u(u-1)$

Where

$\Delta y_k = y_{k+1}- y_k$

$\Delta^2 y_k = y_{k+2} - 2y_{k+1} +y_k$

$u = (x-x_k) / h$

$h = x_{k+1} -x_k$

Also, the area under the parabola is given by

$$I_k = \int_{x_k}^{x_{k+2}} ydx \text{--------------eq1}$$

Since $u = (x-x_k) / h$ and $d(uv) =udv +vdu$

Therefore,

d(hu) = udh+ hdu = $d(hu) = udh +h [d((x-x_k) / h)]$

Also h and x_k are constant, so their differentiation is zero

0+hdu=dx-0

hdu=dx

dx= hdu------------------eq2

Substituting value of eq2 in eq 1 we get

$$I_k = \int_{x_k}^{x_{k+2}} y\ hdu$$

Since $y= y_k +\Delta y_k u$

Also at, u=0 at $x=x_k$ and u=1 at $x= x_{k+1}$

$$I_k = \int_0^2 y\, h du$$

$$I_k = \int_0^2 [\mathrm{y_k} + \Delta \mathrm{y_k} \mathrm{u} + \frac{\Delta^2 \mathrm{y_k}}{2!} \mathrm{u}(\mathrm{u}-1)]\, hdu$$

$$I_k = h[\int_0^2 [\mathrm{y_k} + \Delta \mathrm{y_k} \mathrm{u} + \frac{\Delta^2 \mathrm{y_k}}{2!} \mathrm{u}(\mathrm{u}-1)]\, du]$$

$$I_k = h\, [\mathrm{y_k} \int_0^2 \mathrm{du} + \Delta \mathrm{y_k} \int_0^2 udu + \frac{\Delta^2 \mathrm{y_k}}{2!} \int_0^2 \mathrm{u}\,(\mathrm{u}-1)\mathrm{du}]$$

Since, $\int_a^b dx = x|\,_a^b$

$$I_k = h\, \left[\mathrm{y_k}\, u\Big|_0^2 + \Delta\, \mathrm{y_k}\, \frac{u^2}{2}\Big|_0^2 + \frac{\Delta^2 \mathrm{y_k}}{2!}\, |\frac{\mathrm{u}^3}{3} - \frac{\mathrm{u}^2}{2}|_0^2\right]$$

$$I_k = h\, \left[\mathrm{y_k}(2-0) + \Delta\, \mathrm{y_k} \left(\frac{2^2}{2} - \frac{0^2}{2}\right) + \frac{\Delta^2 \mathrm{y_k}}{2!} (\frac{2^3}{3} - \frac{0^3}{3} - \frac{2^2}{2} + \frac{0^2}{2})\right]$$

$$I_k = h\, \left[\mathrm{y_k}(2) + \Delta\, \mathrm{y_k} * \frac{4}{2} + \frac{\Delta^2 \mathrm{y_k}}{2!} (\frac{8}{3} - \frac{4}{2})\right]$$

$$I_k = h\left[2\mathrm{y_k} + 2\Delta\, \mathrm{y_k} + \frac{4}{3}\, \Delta^2 \mathrm{y_k} - \Delta^2 \mathrm{y_k}\right]$$

$$I_k = h/3[6\mathrm{y_k} + 6\Delta\, \mathrm{y_k} + 4\, \Delta^2 \mathrm{y_k} - 3\Delta^2 \mathrm{y_k}]$$

$$I_k = h/3[6\mathrm{y_k} + 6\Delta\, \mathrm{y_k} + \Delta^2 \mathrm{y_k}]$$

Substituting, $\Delta y_k = y_{k+1} - y_k$ and $\Delta^2 y_k = y_{k+2} - 2y_{k+1} + y_k$

$$I_k = h/3[6\mathrm{y_k} + 6(\mathrm{y_{k+1}} - \mathrm{y_k}) + \mathrm{y_{k+2}} - 2\mathrm{y_{k+1}} + \mathrm{y_k}]$$

$$I_k = h/3[\mathrm{y_k} + 4\mathrm{y_{k+1}} + \mathrm{y_{k+2}}]$$

This is the area under the curve between the points x= x_k and x=x_{k+2}. This covers two intervals of width *‘h’* each. If we use this process repetitively n/2 times, we can get the area under the curve for ‘n’ intervals. Thus, if the range of integration is divided into ‘n’ intervals, each of width h, the required integral over a range x=z and x=b is given by:

$I = I_1 + I_2 + \ldots + I_{n-1}$

$$I = h/3[\mathrm{y_1} + 4\mathrm{y_2} + \mathrm{y_3}] + h/3[\mathrm{y_3} + 4\mathrm{y_4} + \mathrm{y_5}] + \cdots + h/3[\mathrm{y_{n-1}} + 4\mathrm{y_n} + \mathrm{y_{n+1}}]$$

$$I = \frac{h}{3}\left[\mathrm{y_1} + 4\mathrm{y_2} + 2\mathrm{y_3} + \cdots + 4\mathrm{y_n} + \mathrm{y_{n+1}}\right]$$

This is the formula for Simpsons 1/3rd rule. It gives the correct value of the integral only if f(x) is a second order (quadratic) function.

Example

Q: Evaluate $\int_1^2 e^{-\frac{1}{2}x}\,dx$ using four intervals.

Sol: With four intervals, the interval size h = $\frac{2-1}{4}$ = 0.25, the function is calculated as

When x=1, y=f(x) = $e^{-\frac{1}{2}(1)} = 0.607$

When x=1.25, y=f(x) = $e^{-\frac{1}{2}(1.25)} = 0.535$

When x=1.50, y=f(x) = $e^{-\frac{1}{2}(1.50)} = 0.472$

When x=1.75, y=f(x) = $e^{-\frac{1}{2}(1.75)} = 0.417$

When x=2.0, y=f(x) = $e^{-\frac{1}{2}(2.0)} = 0.368$

Therefore, we have the following distribution:

X	1	1.25	1.50	1.75	2.0
f(x)	0.607	0.535	0.472	0.417	0.368

Substituting these values in trapezoidal rule formula

$$I = \frac{h}{3}\left[y_1 + 4y_2 + 2y_3 + 4y_4 + y_5\right]$$

we get,

$$I = \frac{0.25}{3}\left[0.607 + 4(0.535) + 2(0.472) + 4(0.417) + 0.368\right]$$

$$I = 0.477$$

5.2 SIMPSONS 3/8 RULE:

This covers three intervals of width '*h*' each between x=x_k and x=x_{k+3}.

$$I = \frac{3h}{8}[\,y_1 + 3y_2 + 3y_3 + 2y_4 + 3y_5 + 3y_6 + 2y_7 + \cdots + 2y_{n-2} + 3y_{n-1} + 3y_n + y_{n+1}]$$

It gives the correct value of integral only if f(x) is a cubic.

Example

Q: The function f(x) is given as follows:

X	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
Y	1.001	1.008	1.027	1.064	1.125	1.216	1.343	1.512	1.729	2.0

Compute the integral of f(x) between x=0.1 and x=1.0

Sol: Given h=0.1 and n=9
Substituting in Simpson's 3/8 rule, we get:

$I= 3h/8(y_1+3y_2+ 3y_3+2y_4+3y_5+3y_6+2y_7+3y_8+3y_9+y_{10})$

I=3*0.1/8 (1.001+ 3(1.008) + 3(1.027) + 2(1.064) + 3(1.125) + 3(1.216) + 2(1.343) + 3(1.512) + 3(1.729) + 2.0)

I= 1.150

EXERCISE

Q: Programmatic implementation of these methods.

Q: Derivation of 3/8 Simpsons rule.