SIMPLEX METHOD:

Although the graphical method of solving LPP is very efficient in developing the conceptual framework necessary for fully understanding the linear programming process, it suffers from the great limitation that it can handle problems involving only two decision variables. However in real world situations we usually encounter cases, where more than two variables are involved. Thus a new method, known as the simplex method was devised, which can be applied for solving LPPs of any magnitude.

MAXIMIZATION CASE

EXAMPLE:

$$\begin{aligned}
&\text{Maximise} \quad Z = 40x_1 + 35x_2 \\
&\text{subject to} \\
&\qquad 2x_1 + 3x_2 \le 60 \\
&\qquad 4x_1 + 3x_2 \le 96 \\
&\qquad x_1, x_2 \ge 0
\end{aligned}$$

Solution: The various steps involved in obtaining optimal solution using simplex method are as follows:

1. Standardize the problem by converting inequalities of the constraints into equations. This can be done by adding slack variables to each of the constraint equations as shown below:

$$\begin{aligned}
2x_1 + 3x_2 + S_1 &= 60 \\
4x_1 + 3x_2 + S_2 &= 96
\end{aligned}$$

Where S_1 varies from 0 to 60 ($0 \le S_1 \le 60$) and S_2 varies from 0 to 96 ($0 \le S_2 \le 96$).

The objective function also needs modification as it should contain every variable in the system including slack variables or other variables that may be added later. Now the problem can be expressed as:

$$\begin{aligned}
&\text{Maximise} \quad Z = 40x_1 + 35x_2 + 0S_1 + 0S_2 \\
&\text{subject to} \\
&\qquad 2x_1 + 3x_2 + S_1 + 0S_2 = 60 \\
&\qquad 4x_1 + 3x_2 + 0S_1 + S_2 = 96 \\
&\qquad x_1, x_2, S_1, S_2 \ge 0
\end{aligned}$$

2. Obtain the initial tableau and solution.

   Various steps are involved in creating a simplex tableau.

   - List horizontally all the variables contained in the problem.
   - Write all the coefficients involved in the constraint equation under there respective variables.
   - Mention the constraint values on the R.H.S against the rows.
   - Add another row titled c_j into the table which indicates the coefficient of various variables in the objective function.
   - Locate the identity matrix and variables involved in it.
   - Add another row titled "solution" to the table. To determine the solution representing the first feasible solution, set all variables other than those in the identity equal to 0 ad determine values of S_1 and S_2 from the constraint equations.

> *Note: The variables in the identity matrix are known as Basic variables and the remaining are known as non-basic variables.*

   - Add another row titled Δ_j to the table. Δ_j is known as Net-after-opportunity-cost row or Net-evaluation row and is obtained as $\Delta_j = c_j - z_j$.

     To obtain the value of z_j under each variable head column, first each element of that column is multiplied by the corresponding coefficient of the solution variables appearing in the Basis. Then the products are added up to get z_j.

Basis		x_1	x_2	S_1	S_2	b_i
S_1	0	2	3	1	0	60
S_2	0	4	3	0	1	96
c_j		40	35	0	0	
Solution		0	0	60	96	
$\Delta_j = c_j - z_j$		40	35	0	0	

> Note: z_j is obtained as:
>
> For column headed x_1 , $z_1 = 0*2+0*4=0,\quad \Delta_1 = 40-0=40$
>
> For column headed x_2 , $z_2 = 0*3+0*3=0,\quad \Delta_2 = 35-0=35$
>
> For column headed S_1 , $z_3 = 0*1+0*0=0,\quad \Delta_3 = 0-0=0$
>
> For column headed S_2 , $z_4 = 0*0+0*1=0,\quad \Delta_4 = 0-0=0$

3. Optimality Test: A simplex tableau indicates an optimal solution:

   a) If all $\Delta_j \leq 0$, when LPP is of maximization type.

b) If all $\Delta_j \geq 0$, when LPP is of minimization type.

Since in the initial simplex tableau for the above example all $\Delta_j \geq 0$, it suggests that the solution can be improved upon by moving any one of the variables into the solution that are nor there. Thus a revised tableau needs to be generated by following the below mentioned steps until all $\Delta_j \leq 0$.

- Select the variable that has the largest Δ_j value and designate it as the incoming variable (key column).
- Divide b_i values by the corresponding values in the key column, to obtain a new column b_i / a_{ij} , where each entry is known as the replacement ratio.
- Select the row which has the least non negative quotient (also known as the replacement ratio) and designate it as the key row and the variable represents the outgoing variable.
- The intersection of key row and key column is known as the key element and it is marked by a * as depicted in the table.

Basis		x_1	x_2	S_1	S_2	b_i	b_i/a_{ij}
S_1	0	2	3	1	0	60	30
S_2	0	4*	3	0	1	96	24
c_j		40	35	0	0		
Solution		0	0	60	96		
$\Delta_j = c_j - z_j$		40	35	0	0		

Key row

Key column

Using the key row, key column and key element information another simplex tableau is derived wherein various elements are obtained as follows.

- Divide each element of the key row by the key element to get corresponding values in the new table. This derived row is known as the replacement row.
- For each row other than the key row,

$$New\ row\ element = Old\ row\ element - (Row\ element\ in\ the\ key\ column * \\ Corresponding\ replacement\ row\ value)$$

The revised simplex table using the above steps is depicted below:

Basis		x_1	x_2	S_1	S_2	b_i	b_i/a_{ij}
S_1	0	0	3/2*	1	-1/2	12	8
x_1	40	1	3/4	0	¼	24	32
c_j		40	35	0	0		
Solution		24	0	12	0		
$\Delta_j = c_j - z_j$		0	5	0	-10		

Key row

Key column

Since all Δ_j not less than equal to 0, another revised simplex table is generated using the above mentioned steps. The table so obtained is depicted below.

Basis		x_1	x_2	S_1	S_2	b_i
x_2	35	0	1	2/3	-1/3	8
x_1	40	1	0	-1/2	1/2	18
c_j		40	35	0	0	
Solution		18	8	0	0	
$\Delta_j = c_j - z_j$		0	0	-10/3	-25/3	

Since all $\Delta_j \leq 0$, $x_1 = 18$ and $x_2 = 8$ corresponds to the optimal solution.

LIMITATIONS TO SIMPLEX METHOD

Applying the simplex method with only slack variables added is not sufficient for LPP that have the objective function of minimization type.

Example:

$$\begin{aligned} &\textit{Minimize} && Z = 40x_1 + 24x_2 \\ &\textit{subject to} && 20x_1 + 50x_2 \geq 4800 \\ & && 80x_1 + 50x_2 \geq 7200 \\ & && x_1, x_2 \geq 0 \end{aligned}$$

We first standardize the problem by converting all inequalities into equations by adding slack variables as shown below.

$$\begin{aligned}
&Minimize && Z = 40x_1 + 24x_2 + 0S_1 + 0S_2 \\
&subject\ to && 20x_1 + 50x_2 - S_1 = 4800 \\
& && 80x_1 + 50x_2 - S_2 \\
& && x_1, x_2, S_1, S_2 \geq 0
\end{aligned}$$

We know that the simplex method needs an initial solution to get the process started. In this case an initial solution does not exist because if we let x_1 and x_2 each equal to 0, we get S_1 =-4800 and S_2 =-7200 which is not feasible as it violates the non negativity constraint. Also in terms of simplex tableau when we write all the information in the tables, we do not get an identity because the coefficient values of surplus variables is -1. To overcome this problem a new method called Big-M method was devised.

BIG-M METHOD

Big-M method is a variant of simplex method which is used to solve problems where an identity is not obtained. In this method, we add artificial variables into the model to obtain an initial solution. Unlike slack or surplus variables, artificial variables have no tangible relationship with the decision problem. Their sole purpose is to provide an initial solution to the given problem.

Since artificial variables do not represent any quantity relating to the decision problem, they must be driven out of the system and must not show in the final solution (and if at all they do, it represents a situation of infeasibility). This can be ensured by assigning an extremely high cost to them. Generally a value M is assigned to each artificial variable, where M represents a number higher than any finite number. For this reason, this method of solving the problems where artificial variables are involved is termed as the Big-M method. When the problem is of minimization nature, we assign in the objective function a coefficient of +M to each of the artificial variables. On the other hand, for problems with objective function of maximization type, each of the artificial variables introduced has a coefficient of -1.

EXAMPLE:

$$\begin{aligned}
&Minimize && Z = 40x_1 + 24x_2 && total\ cost \\
&subject\ to && 20x_1 + 50x_2 \geq 4800 && phosphate\ requirement \\
& && 80x_1 + 50x_2 \geq 7200 && nitrogen\ requirement \\
& && x_1, x_2 \geq 0
\end{aligned}$$

Solution:

First of all we standardize the problem by adding necessary slack and artificial variables such that the problem looks as shown below:

$$\begin{aligned} &Minimize && Z = 40x_1 + 24x_2 + 0S_1 + 0S_2 + MA_1 + MA_2 \\ &subject\ to && 20x_1 + 50x_2 - S_1 + A_1 = 4800 \\ & && 80x_1 + 50x_2 - S_2 + A_2 = 7200 \\ & && x_1, x_2, S_1, S_2, A_1, A_2 \geq 0 \end{aligned}$$

Next step is to create the initial simplex tableau by listing all the variables involved in the problem horizontally as discusses above. The resulting initial simplex tableau is depicted below:

Basis		x_1	x_2	S_1	S_2	A_1	A_2	b_i
A_1	M	20	50	-1	0	1	0	4800
A_2	M	80	50	0	-1	0	1	7200
c_j		40	24	0	0	M	M	
Solution		0	0	0	0	4800	7200	
$\Delta_j = c_j - z_j$		40-100M	24-100M	M	M	0	0	

Note: z_j is obtained as:

For column headed x_1 , z_1 = $20*M+80*M=100M, \quad \Delta_1 = 40-100M$

For column headed x_2 , z_2 = $50*M+50*M=100M, \quad \Delta_2 = 24-100M$

For column headed S_1 , z_3 = $-1*M+0*M=-M, \quad \Delta_3 = 0-(-M)=M$

For column headed S_2 , z_4 = $0*M+(-1*M)=-M, \quad \Delta_4 = 0-(-M)=M$

For column headed A_1 , z_5 = $1*M+0*M=M, \quad \Delta_5 = M-M=0$

For column headed A_2 , z_6 = $0*M+1*M=M, \quad \Delta_6 = M-M=0$

For the minimization problem, the optimal solution is indicated when the values in Δ_j row are 0 or positive. The presence of negative Δ_j values indicates that the solution can be improved.

To obtain a revised simplex tableau, the incoming variable is selected to be the one with the most negative Δ_j value and the column headed by this variable is called the key column. The row which has the least non negative quotient becomes the key row. Finally the revised simplex tableau is generated the same way as done previously for the maximization case as shown below.

Basis		x_1	x_2	S_1	S_2	A_1	A_2	b_i	b_i/a_{ij}
A_1	M	20	50*	-1	0	1	0	4800	96
A_2	M	80	50	0	-1	0	1	7200	144
c_j		40	24	0	0	M	M		
Solution		0	0	0	0	4800	7200		
$\Delta_j = c_j - z_j$		40-100M	24-100M	M	M	0	0		

Key row

Key column

Using the key row, key column and key element information another simplex tableau is derived wherein various elements are obtained the same way as done for the maximization case thereby giving us another revised simplex tableau.

Basis		x_1	x_2	S_1	S_2	A_1	A_2	b_i	b_i/a_{ij}
x_2	24	2/5	1	-1/50	0	1/50	0	96	240
A_2	M	60*	0	1	-1	-1	1	2400	40
c_j		40	24	0	0	M	M		
Solution		0	96	0	0	0	2400		
$\Delta_j = c_j - z_j$		$\frac{152}{5} - 60M$	0	$\frac{12}{25} - M$	M	$2M - \frac{12}{25}$	0		

Key row

Key column

This process of generating revised simplex tableau continues till negative Δ_j values are present.

Basis		x_1	x_2	S_1	S_2	A_1	A_2	b_i	b_i/a_{ij}
x_2	24	0	1	-2/75	1/150	2/75	-1/150	80	-3000
x_1	40	1	0	1/60*	-1/60	-1/60	1/60	40	2400
c_j		40	24	0	0	M	M		
Solution		40	80	0	0	0	0		
$\Delta_j = c_j - z_j$		0	0	-2/75	38/75	$M+\frac{2}{75}$	$M+\frac{38}{75}$		

Key row

Key column

The presence of negative Δ_j value in S_1 means the solution needs further improvement.

Basis		x_1	x_2	S_1	S_2	A_1	A_2	b_i
x_2	24	8/5	1	0	-1/50	0	1/50	144
S_1	0	60	0	1	-1	-1	1	2400
c_j		40	24	0	0	M	M	
Solution		0	144	2400	0	0	0	
$\Delta_j = c_j - z_j$		8/5	0	0	12/25	M	$M+\frac{12}{25}$	

Since all $\Delta_j \geq 0$, $x_2 = 144$ and $S_1 = 2400$ corresponds to the optimal solution.

The value of $S_1 = 2400$ indicates the surplus phosphate ingredient obtained by buying the least cost mix.

EXAMPLE:

$$\begin{aligned} &\textit{Maximize} && Z = 2x_1 + 4x_2 \\ &\textit{subject to} && 2x_1 + x_2 \leq 18 \\ & && 3x_1 + 2x_2 \geq 30 \\ & && x_1 + 2x_2 = 26 \\ & && x_1, x_2 \geq 0 \end{aligned}$$

Solution:

First of all we standardize the problem by adding necessary slack and artificial variables such that the problem looks as shown below:

$$
\begin{aligned}
&\textit{Maximize} && Z = 2x_1 + 4x_2 + 0S_1 + 0S_2 - MA_1 - MA_2 \\
&\textit{subject to} && 2x_1 + x_2 + S_1 = 18 \\
& && 3x_1 + 2x_2 - S_2 + A_1 = 30 \\
& && x_1 + 2x_2 + A_2 = 26 \\
& && x_1, x_2, S_1, S_2, A_1, A_2 \geq 0
\end{aligned}
$$

The solution is combined in the following tables.

Basis		x_1	x_2	S_1	S_2	A_1	A_2	b_i
S_1	0	2	1	1	0	0	0	18
A_1	-M	3	2	0	-1	1	0	30
A_2	-M	1	2	0	0	0	1	26
c_j		2	4	0	0	-M	-M	
Solution		0	0	18	0	30	26	
$\Delta_j = c_j - z_j$		4M+2	4M+4	0	-M	0	0	

For the maximization problem, the optimal solution is indicated when the values in Δ_j row are 0 or negative. The presence of positive Δ_j values indicates that the solution can be improved.

To obtain a revised simplex tableau, the incoming variable is selected to be the one with the largest Δ_j value and the column headed by this variable is called the key column. The row which has the least non negative quotient becomes the key row. Finally the revised simplex tableau is generated the same way as done previously for the maximization case as shown below.

Basis		x_1	x_2	S_1	S_2	A_1	A_2	b_i	b_i/a_{ij}
S_1	0	2	1	1	0	0	0	18	18
A_1	-M	3	2	0	-1	1	0	30	15
A_2	-M	1	2*	0	0	0	1	26	13
c_j		2	4	0	0	-M	-M		
Solution		0	0	18	0	30	26		
$\Delta_j = c_j - z_j$		4M+2	4M+4	0	-M	0	0		

Key row

Key column

Basis		x_1	x_2	S_1	S_2	A_1	A_2	b_i	b_i/a_{ij}
S_1	0	3/2	0	1	0	0	-1/2	5	10/3
A_1	-M	2*	0	0	-1	1	-1	4	2
x_2	4	1/2	1	0	0	0	1/2	13	26
c_j		2	4	0	0	-M	-M		
Solution		0	13	5	0	4	0		
$\Delta_j = c_j - z_j$		2M	0	0	-M	0	-2-2M		

Key row

Key column

Basis		x_1	x_2	S_1	S_2	A_1	A_2	b_i
S_1	0	0	0	1	3/4	-3/4	1/4	2
x_1	2	1	0	0	-1/2	1/2	-1/2	2
x_2	4	0	1	0	1/4	-1/4	3/4	12
c_j		2	4	0	0	-M	-M	
Solution		2	12	2	0	0	0	
$\Delta_j = c_j - z_j$		0	0	0	0	-M	-M-2	

The last table depicts that all $\Delta_j \leq 0$ which indicates this solution of $x_1 = 2$ and $x_2 = 12$, $S_1 = 2$ and other variables = 0 is an optimal solution.

TWO PHASE METHOD:

This method is an alternative to Big-M method and is so called because it separates the solution procedure into two phases. In phase 1 all the artificial variables are eliminated from the basis. If a feasible solution is obtained in this phase, which has no artificial variables in the basis in the final tableau, then we proceed to phase II. In this phase we use the solution from phase I as the initial basic feasible solution and use simplex method to determine the optimal solution.

Phase I:

The various steps involved in this phase are:

1. Convert each of the constraints into equality relationships by subtracting a surplus variable and then add an artificial variable.
2. Assign zero coefficients to each of the primary (x_j) and the surplus variables. Also assign unit coefficients to each of the artificial variables (in a maximization problem, the

coefficients to each of the artificial variables shall be -1). This amounts to replacing the objective function of the original problem by the sum of artificial variables.

3. Solve the auxiliary problem by applying the simplex method. If the original problem has a feasible solution, then this problem shall have an optimal solution with optimal value of the objective function equal to zero as each of the artificial variables $A_1, A_2, \ldots, A_n$ shall be equal to 0. The simplex method would remove all the artificial variables from the basis one by one.

Phase II:

In this phase, start with the optimal solution contained in the final simplex tableau of phase I. Remove c_j row values at the bottom of the optimal tableau and replace them with c_j values of the original problem. Furthermore, eliminate the entries in the columns headed $A_1, A_2, \ldots, A_n$. Apply simplex algorithm to the problem contained in the new tableau to obtain the optimal solution.

EXAMPLE:

$$\begin{array}{lll} \textit{Minimize} & Z = 40x_1 + 24x_2 & \textit{total cost} \\ \textit{subject to} & 20x_1 + 50x_2 \geq 4800 & \textit{phosphate requirement} \\ & 80x_1 + 50x_2 \geq 7200 & \textit{nitrogen requirement} \\ & x_1, x_2 \geq 0 & \end{array}$$

Solution:

Phase I: First we introduce surplus and artificial variables, and then rewrite the objective function by assigning a 0 coefficient to the decision variables and a coefficient 1 to the artificial variables. The problem becomes:

$$\begin{array}{ll} \textit{Minimize} & Z = 0x_1 + 0x_2 + 0S_1 + 0S_2 + A_1 + A_2 \\ \textit{subject to} & 20x_1 + 50x_2 - S_1 + A_1 = 4800 \\ & 80x_1 + 50x_2 - S_2 + A_2 = 7200 \\ & x_1, x_2, S_1, S_2, A_1, A_2 \geq 0 \end{array}$$

Now we implement the simplex method to eliminate the artificial variables as shown in the following tables.

Basis		x_1	x_2	S_1	S_2	A_1	A_2	b_i
A_1	1	20	50	-1	0	1	0	4800
A_2	1	80	50	0	-1	0	1	7200
c_j		0	0	0	0	1	1	
Solution		0	0	0	0	4800	7200	
$\Delta_j = c_j - z_j$		-100	-100	1	1	0	0	

Basis		x_1	x_2	S_1	S_2	A_1	A_2	b_i	b_i/a_{ij}
A_1	1	20	50	-1	0	1	0	4800	240
A_2	1	80*	50	0	-1	0	1	7200	90
c_j		0	0	0	0	1	1		
Solution		0	0	0	0	4800	7200		
$\Delta_j = c_j - z_j$		-100	-100	1	1	0	0		

Key row

Key column

Basis		x_1	x_2	S_1	S_2	A_1	A_2	b_i	b_i/a_{ij}
A_1	1	0	75/2*	-1	1/4	1	-1/4	3000	80
x_1	0	1	5/8	0	-1/80	0	1/80	90	144
c_j		0	0	0	0	1	1		
Solution		90	0	0	0	3000	0		
$\Delta_j = c_j - z_j$		0	-75/2	0	-1/4	0	5/4		

Key row

Key column

Basis		x_1	x_2	S_1	S_2	A_1	A_2	b_i
x_2	0	0	1	-2/75	1/150	2/75	-1/150	80
x_1	0	1	0	1/60	-1/60	-1/60	-1/60	40
c_j		0	0	0	0	1	1	
Solution		40	80	0	0	0	0	
$\Delta_j = c_j - z_j$		0	0	0	0	1	1	

Since all $\Delta_j \geq 0$ and the optimal value of the standardized objective function including the artificial variables equals 0, feasible solution exists and we can proceed to phase II.

Phase II:

Reproduce the above table by replacing c_j row by the respective coefficients from the objective function of the original problem and delete columns A_1 and A_2. The new table obtained is shown below and is solved using simplex method till the optimal solution is obtained (i.e. all $\Delta_j \geq 0$).

Basis		x_1	x_2	S_1	S_2	b_i	b_i/a_{ij}
x_2	24	0	1	-2/75	1/150	80	-3000
x_1	40	1	0	1/60*	-1/60	40	2400
c_j		40	24	0	0		
Solution		40	80	0	0		
$\Delta_j = c_j - z_j$		0	0	-2/75	38/75		

Key row

Key column

Basis		x_1	x_2	S_1	S_2	b_i
x_2	24	8/5	1	0	-1/50	144
S_1	0	60	0	1	-1	2400
c_j		40	24	0	0	
Solution		0	144	2400	0	
$\Delta_j = c_j - z_j$		8/5	0	0	12/25	

The optimal solution is x_1=0 and x_2=144.

SPECIAL CASES:

- **MULTIPLE OPTIMAL SOLUTIONS**
  In every simplex tableau generated when solving various problems, the basic variables have all $\Delta_j = 0$, but not the non basic ones. When a solution is indicated to be optimal and the Δ_j value for none of the non basic variables is 0, the solution is unique in the

sense that no other solution to the given problem exists. However, when a non basic variable in an optimal solution has a 0 value for Δ_j, then multiple optimal solutions exist.

Example:

$$\begin{aligned} &\textit{Maximixe} && Z = 8x_1 + 16x_2 \\ &\textit{subject to} && x_1 + x_2 \le 200 \\ & && x_2 \le 125 \\ & && 3x_1 + 6x_2 \le 900 \\ & && x_1, x_2 \ge 0 \end{aligned}$$

The final simplex tableau for the above problem depicting the optimal solution is as shown below:

Basis		x_1	x_2	S_1	S_2	S_3	b_i
S_1	0	0	0	1	1	-1/3	25
x_2	16	0	1	0	1	0	125
x_1	8	1	0	0	-2	1/3	50
c_j		8	16	0	0	0	
Solution		50	125	25	0	0	
$\Delta_j = c_j - z_j$		0	0	0	0	-8/3	

As evident from the above table, not only the basic variables S_1, x_2 and x_1 have $\Delta_j = 0$ but also the non basic variable S_2 has $\Delta_j = 0$, thereby indicating that multiple optimal solutions exist for this problem.

- **INFEASIBILITY:**

  Infeasibility is said to exist when a given problem has no feasible solution. In terms of the simplex method, when in the final solution, an artificial variable is in the basis at a positive value then there is no feasible solution to the problem.

Example:

$$\begin{aligned} &\textit{Maximixe} && Z = 20x_1 + 30x_2 \\ &\textit{subject to} && 2x_1 + x_2 \le 40 \\ & && 4x_1 - x_2 \le 20 \\ & && x_1 \ge 30 \\ & && x_1, x_2 \ge 0 \end{aligned}$$

The final simplex tableau obtained for the above problem, when all $\Delta_j \le 0$ is as shown below:

Basis		x_1	x_2	S_1	S_2	S_3	A_1	b_i
x_2	30	0	1	2/3	-1/3	0	0	20
x_1	20	1	0	1/6	1/6	0	0	10
A_1	-M	0	0	-1/6	-1/6	-1	1	20
c_j		20	30	0	0	0	-M	
Solution		10	20	0	0	0	20	
$\Delta_j = c_j - z_j$		0	0	$-\frac{70}{3}-\frac{M}{6}$	$\frac{20}{3}-\frac{M}{6}$	-M	0	

In the above table although all $\Delta_j \le 0$, but an artificial variable is present in the basis and it has a positive value in the solution row thereby indicating that no feasible solution exists to the given problem.

In terms of two phase method, infeasibility if present is detected in phase I itself. In such a case the objective function value is not equal to zero (it is positive for the minimization and negative for the maximization problems) and one artificial variable is found in the basis in the final table of phase I as depicted in the below mentioned example.

Example:

$$\begin{aligned} &\textit{Maximixe} && Z = 20x_1 + 30x_2 \\ &\textit{subject to} && 2x_1 + x_2 \le 40 \\ &&& 4x_1 - x_2 \le 20 \\ &&& x_1 \ge 30 \\ &&& x_1, x_2 \ge 0 \end{aligned}$$

After standardizing the problem by adding the necessary surplus and artificial variables, the auxiliary problem looks as follows:

$$\begin{aligned} &\textit{Maximixe} && Y = 0x_1 + 0x_2 + 0S_1 + 0S_2 + 0S_3 - A_1 \\ &\textit{subject to} && 2x_1 + x_2 + S_1 = 40 \\ &&& 4x_1 - x_2 + S_2 = 20 \\ &&& x_1 - S_3 + A_1 = 30 \\ &&& x_1, x_2, S_1, S_2, S_3, A_1 \ge 0 \end{aligned}$$

The final simplex tableau obtained at the end of phase I is depicted below:

Basis		x_1	x_2	S_1	S_2	S_3	A_1	b_i
x_2	0	0	1	2/3	-1/3	0	0	20
x_1	0	1	0	1/6	1/6	0	0	10
A_1	-1	0	0	-1/6	-1/6	-1	1	20
c_j		0	0	0	0	0	-1	
Solution		10	20	0	0	0	20	
$\Delta_j = c_j - z_j$		0	0	-1/6	-1/6	-1	0	

At the end of phase I the objective function value is -20 and artificial variable A_1 is in the basis at a positive level, at 20.

> *Note: The objective function value =-20 is obtained by substituting the values from the final simplex tableau of phase I in the auxiliary problem:*
>
> $$Y = 0x_1 + 0x_2 + 0S_1 + 0S_2 + 0S_3 - A_1$$

- **UNBOUNDEDNESS**
  LPP is said to have an unbounded solution if its objective function value can be increased (in case of a maximization problem) or decreased (in case of a minimization problem) without limit. In terms of simplex method a problem is said to have an unbounded solution if the value of all the replacement ratios in a particular simplex tableau is negative or infinity in which case the algorithm terminates. The following problem depicts an unbounded solution:

$$\begin{aligned} &\textit{Maximixe} && Z = 10x_1 + 20x_2 \\ &\textit{subject to} && 2x_1 + 4x_2 \geq 40 \\ &&& x_1 + 5x_2 \geq 15 \\ &&& x_1, x_2 \geq 0 \end{aligned}$$

After standardizing the problem by adding the necessary surplus and artificial variables the problem becomes:

$$\begin{aligned} &\textit{Maximixe} && Z = 10x_1 + 20x_2 + 0S_1 + 0S_2 - MA_1 - MA_2 \\ &\textit{subject to} && 2x_1 + 4x_2 - S_1 + A_1 = 16 \\ &&& x_1 + 5x_2 - S_2 + A_2 = 15 \\ &&& x_1, x_2, S_1, S_2, A_1, A_2 \geq 0 \end{aligned}$$

The final simplex tableau obtained after generating a series of simplex tableaus is given below and represents and unbounded solution as all the replacement ratios (i.e. b_i/a_{ij}) are negative.

Basis		x_1	x_2	S_1	S_2	A_1	A_2	b_i	b_i/a_{ij}
x_1	10	1	5	0	-1	0	1	15	-15
S_1	0	0	6	1	-2	-1	2	14	-7
c_j		10	20	0	0	-M	-M		
Solution		15	0	14	0	0	0		
$\Delta_j = c_j - z_j$		0	-30	0	10	-M	-M-10		

- **DEGENERACY:**

Degeneracy in LPP occurs when one or more of the basic variable assumes 0 value. We know that for n-variable, m-constraint problem, there would be m basic and nm non basic variables and the basic variables would assume positive values. However in case a basic variable assumes a value of 0, then that variable and the solution are labeled as degenerate. Thus in condition of degeneracy, the solution would contain a smaller number of non-zero variables than the number of constraints.

Example:

$$\begin{aligned} &\textit{Maximixe} && Z = 28x_1 + 30x_2 \\ &\textit{subject to} && 6x_1 + 3x_2 \le 18 \\ & && 3x_1 + x_2 \le 8 \\ & && 4x_1 + 5x_2 \le 30 \\ & && x_1, x_2 \ge 0 \end{aligned}$$

After standardization of the problem the initial simplex tableau gives duplicate values for the replacement ratios as depicted in the following table.

$$\begin{aligned} &\textit{Maximixe} && Z = 28x_1 + 30x_2 + 0S_1 + 0S_2 + 0S_3 \\ &\textit{subject to} && 6x_1 + 3x_2 + S_1 = 18 \\ & && 3x_1 + x_2 + S_2 = 8 \\ & && 4x_1 + 5x_2 + S_3 = 30 \\ & && x_1, x_2, S_1, S_2, S_3 \ge 0 \end{aligned}$$

Basis		x_1	x_2	S_1	S_2	S_3	b_i	b_i/a_{ij}	
S_1	0	6	3	1	0	0	18	6	←
S_2	0	3	1	0	1	0	8	8	Tie
S_3	0	4	5	0	0	1	30	6	←
c_j		28	30	0	0	0			
Solution		0	0	18	8	30			
$\Delta_j = c_j - z_j$		28	30	0	0	0			
			↑						

As there is a tie between the first and third rows, either of the variables S_1 and S_3 could be taken to be the outgoing variable. By taking S_1 as the outgoing variable, the revised tableau is as under:

Basis		x_1	x_2	S_1	S_2	S_3	b_i
x_2	30	2	1	1/3	0	0	6
S_2	0	1	0	-1/3	1	0	2
S_3	0	-6	0	-5/3	0	1	0
c_j		28	30	0	0	0	
Solution		0	6	0	2	0	
$\Delta_j = c_j - z_j$		-32	0	-10	0	0	

This tableau gives the optimal solution x_1=0 and x_2=6, with objective function value =180. By taking S_3 as the outgoing variable, the revised tableau is as under:

Basis		x_1	x_2	S_1	S_2	S_3	b_i	b_i/a_{ij}	
S_1	0	18/5*	0	1	0	-3/5	0	0	←
S_2	0	11/5	0	0	1	-1/5	2	10/11	
x_2	30	4/5	1	0	0	1/5	6	15/2	
c_j		28	30	0	0	0			
Solution		0	6	0	2	0			
$\Delta_j = c_j - z_j$		4	0	0	0	-6			
		↑							

The above table does not represent an optimal solution as all Δ_j are not less than or equal to 0. The revised simplex tableau gives us the optimal solution as shown below:

Basis		x_1	x_2	S_1	S_2	S_3	b_i
x_1	28	1	0	5/18	0	-1/6	0
S_2	0	0	0	-11/18	1	1/6	2
x_2	30	0	1	-2/9	0	1/3	6
c_j		28	30	0	0	0	
Solution		0	6	0	2	0	
$\Delta_j = c_j - z_j$		0	0	-10/9	0	-16/3	

Various observations from the above example:

a) In the optimal solution simplex table obtained by using S_1 as the outgoing variable, the basis variable S_3 has 0 value in the solution row thereby making S_3 and the solution in this case as degenerate.

b) In the optimal solution simplex table obtained by using S_3 as the outgoing variable, the basis variable x_1 has 0 value in the solution row thereby making x_1 and the solution in this case as degenerate.

c) Whenever there is a tie in the replacement ratios for determining outgoing variable, the next tableau would give a degenerate solution.

d) We know that successive simplex tableaus represent improvements in the value of the objective function (increase for the maximization problem and decrease for minimization problem). However, when the outgoing variable happens to be a degenerate variable, the objective function value in the next tableau does not change as is evident from the last two tables of the above example.

PRACTICE QUESTIONS

1) Solve the following LPP:

$$\begin{aligned} &\textit{Maximixe} && Z = 8x_1 - 4x_2 \\ &\textit{subject to} && 4x_1 + 5x_2 \leq 20 \\ & && -x_1 + 3x_2 \geq -23 \\ & && x_1 \geq 0, x_2 \quad \textit{unrestricted in sign} \end{aligned}$$

2) Solve the following LPP:

$$\begin{aligned} \textit{Maximixe} \quad & Z = 6x_1 + 20x_2 \\ \textit{subject to} \quad & 2x_1 + x_2 \le 32 \\ & 3x_1 + 4x_2 \le 80 \\ & x_1 \ge 8 \\ & x_2 \ge 10 \end{aligned}$$

3) A finished product must weigh exactly 150grams. The two raw materials used for manufacturing the product are A, with a cost of Rs 2 per unit and B with a cost of Rs 8 per unit. At least 14 units of B and not more than 20 units of A must be used. Each unit of A and B weighs 5 and 10 grams respectively.
How much of each type of raw material should be used for each unit of the final product in order to minimize the cost? Use simplex method.

4) A firm uses three machines in the manufacture of three products. Each unit of product A requires 3 hours on machine I, 2 hours on machine II and 1 hour on machine III. Each unit of product B requires 4 hours on machine I, 1 hour on machine II and 3 hours on machine III, while each unit of product C requires two hours on each of the three machines. The contribution margin of the three products is Rs 30, Rs 40 and Rs 35 per unit respectively. The machine hours available on three machines are 90, 54 and 93 respectively.

a) Formulate the above problem as LPP.
b) Obtain optimal solution to the problem by using the simplex method. Which of the three products shall not be produced by the firm? Why?
c) Calculate the percentage of capacity utilization in the optimal solution.
d) What are the shadow prices of the machine hours.
e) Is the optimal solution degenerate?