GAUSS ELIMINATION WITH PIVOTING….

In the basic gauss elimination method, the element a_{ij} when i=j is known as a pivot element. Each row is normalised by dividing the coeffients of that row by its pivot element. That is,

$$a_{kj} = \frac{a_{kj}}{a_{kk}} \quad j=1, \ \ldots, n.$$

If $a_{kk}=0$, k^{th} row cannot be normalised. Therefore, the procedure fails.

One way to overcome this problem is to interchange this row with another row below it which does not have a zero element in that position.

For e.g;

Solve the following 3*3 system using the basic gauss elimination method

$$3x_1+6x_2+x_3 = 16$$

$$2x_1+4x_2+3x_3= 13$$

$$x_1+3x_2+2x_3= 9$$

After the first step of elimination using multiplication factor 2/3 and 1/3 , we obtain the new system as follows:

$$3x_1+6x_{2+}x_3=16$$

$$0+0+7x_3=7$$

$$0+3x_2+5x_3=11$$

At this point, $a_{22}=0$ and , therefore, the elimination procedure breaks down. We need to reorder the equation as shown below:

$$3x_1+6x_2+x_3=16$$

$$3x_2+5x_3=11$$

$$7x_3=7$$

The process of elimination is complete and the solution is :

$x_3=1$, $x_2=2$, and $x_1=1$.

The reordering of the rows is done such that a_{kk} of the row to be normalised is not zero. There may be more than one non-zero values in the k^{th} column below the a_{kk}. It is suggested that the row with zero pivot element should be interchanged with the row having the largest absolute value coefficient in that position.

It can be proved that round-off errors would be reduced if the absolute value of the pivot element is large.

In general, the reordering of equations is done to improve accuracy, even if the pivot element is not zero.

The procedure of reordering involves the following steps:

1. Search and locate the largest absolute value among the coefficients in the first column.
2. Exchange the first row with the row containing that element.
3. Then eliminate the first variable in the second equation as explained earlier.
4. When the second row becomes the pivot row, search for the coefficients in the second column from the second row to the n^{th} row and locate the largest coefficient. Exchange the second row with the row containing the large coeffient.
5. Continue this procedure till (n-1) unknowns are eliminated.

This process is referred to as partial pivoting.

There is an alternative scheme known as complete pivoting in which, at each stage, the largest element in any of the remaining rows is used as the pivot. Complete pivoting requires a lot of overhead and , therefore, it is not generally used (though it may yield slightly improved numerical stability).

ALGORITHM:

1. Input n, a_{ij} and b_i values.
2. Beginning from the first equation,

i. Check for the pivot element.
ii. If it is the largest among the elements below it, obtain the derived system.
iii. Otherwise identify the largest element and make it the pivot element.
iv. Interchange , the original pivot equation with the one containing the largest element so that, the later becomes the new pivot equation.
v. Obtain the derived system.
vi. Continue the process till the system is reduced to triangular form.

3. Compute x_i values by back substitution.
4. Print results.

Q: Solve the following system of equations using gauss elimination with pivoting?

$$2x_1+2x_2+x_3=6$$

$$4x_1+2x_2+3x_3=4$$

$$x_1-x_2+x_3=0$$

Sol, Here, we have

$$2x_1+2x_2+x_3=6 \qquad (1)$$

$$4x_1+2x_2+3x_3=4 \qquad (2)$$

$$x_1-x_2+x_3=0 \qquad (3)$$

Step 1: Interchanging

$$4x_1+2x_2+3x_3=4 \qquad (4)$$

$$2x_1+2x_2+x_3=6 \qquad (5)$$

$$x_1-x_2+x_3=0 \qquad (6)$$

Step 2: Using elimination

$$\frac{2}{4}\ (4x_1+2x_2+3x_3\ ) = \frac{2}{4}\ (4)$$

$$= \ \frac{1}{2}\ (4x_1+ 2x_2+3x_3= 2$$

$$2x_1+x_2+\frac{3}{2}\ x_3=2$$

Subtracting the above equation from equation(5) ,we get,

$$2x_1+2x_2+x_3=6$$

$$-(\ 2x_1 +x_2 + \frac{3}{2}x_3)= -2$$

$$=> x_2 - \frac{1}{2} x_3= 4 \qquad (7)$$

Also, $$\frac{1}{4} (4x_1+2x_2+3x_3) = \frac{1}{4}(4)$$

$$x_1+ \frac{1}{2} x_2+ \frac{3}{4} x_3 =1$$

Subtracting above equation from equation 6, we have:

$$x_1 -x_2+x_3=0$$

$$-(x_1+ \frac{1}{2} x_2+ \frac{3}{4} x_3 =1$$

$$=> \quad - \frac{3}{2} x_2+ \frac{1}{4}x_3= -1 \qquad (8)$$

After step 2, the new set of equation is:

$$4x_1+2x_2+3x_3=4 \qquad (9)$$

$$x_2- \frac{1}{2} x_3=4 \qquad (10)$$

$$- \frac{3}{2}x_2+ \frac{1}{4} x_3= -1 \qquad (11)$$

Step 3: Interchanging equation 10 and 11, we get

$$4x_1+2x_2+3x_3=4 \qquad (12)$$

$$- \frac{3}{2} x_2+ \frac{1}{4}x_3= -1 \qquad (13)$$

$$X_2- \frac{1}{2}x_3=4 \qquad (14)$$

Step4: Elimination of x_2 from equation 14.

$$- \frac{2}{3}(-\frac{3}{2} x_2+ \frac{1}{4} x_3) = - \frac{2}{3} (-1)$$

$$x_2 - \frac{1}{6} x_3 = \frac{2}{3}$$

Subtracting above equation from equation 14,we have

$$x_2 - \frac{1}{2}x_3 = 4$$

$$=> -\frac{2}{6}x_3 = \frac{10}{3}$$

$$=> -\frac{1}{3}x_3 = \frac{10}{3} \qquad (15)$$

After Step 4, the new set of equation becomes:

$$4x_1+2x_2+3x_3=4 \qquad (16)$$

$$-\frac{3}{2}x_2+\frac{1}{4}x_3 = -1 \qquad (17)$$

$$-\frac{1}{3}x_3 = \frac{10}{3} \qquad (18)$$

Step 5: Using backward substitution, we have

$$-\frac{1}{3}x_3 = \frac{10}{3}$$

$$x_3 = -10$$

Substituting value of x_3 in equation 17, we have

- $\frac{3}{2}x_2 + \frac{1}{4}x_3 = -1$
- $\frac{3}{2}x_2 + \frac{1}{4}(-10) = -1$
- $\frac{3}{2}x_2 - \frac{5}{2} = -1$
- $\frac{3}{2}x_2 = -1 + \frac{5}{2}$
- $\frac{3}{2}x_2 = \frac{3}{2}$
- $x_2 = \frac{3}{2} + \frac{2}{3}$
- $x_2 = 1$

$$=> x_2 = -1$$

Substituting values of x_2 and x_3 in equation 16, we have

$$4x_1+2x_2+3x_3=4$$

$$4x_1+2(-1) + 3(-10)=4$$

$$4x_1 - 2 - 30=4$$

$4x_1-32=4$

$4x_1=4+32$

$4x_1=36$

$x_1=9$

Thus the solution is,

$x_1 = 9$, $x_2 = -1$ and $x_3 = -10$.

LU DECOMPOSITION OR TRIANGULAR FACTORISATION METHOD…

The coefficient matrix A of a system of linear equations can be factorised (or decomposed) into two triangular matrices L and U such that,

$$A=LU$$

Where,

$$L=\begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{n1} & l_{n2} & l_{nn} \end{bmatrix}$$

And

$$U=\begin{bmatrix} u_{11} & u_{12} & u_{nn} \\ 0 & u_{22} & u_{2n} \\ 0 & 0 & u_{nn} \end{bmatrix}$$

L is known as lower triangular matrix and U is known as upper triangular matrix.

Once A is factorised into L and U, the system of equations

$$Ax=b$$

Can be expressed as follows:

$$(LU)x = b$$

Or $L(Ux) = b$

Let us assume that

$Ux = z$

Where z is an unknown vector. Substituting in above equation we get

$Lz = b$

Now, we can solve that system

$Ax=b$ in two stages:

1. Solve the equation
   $Lz = b$
   For z by forward substitution.
2. Solve the equation
   $Ux = z$

For x using z found in stage 1 by back substitution.

The decomposition with L having unit diagonal values is called the Dolittle LU decomposition while the other one with U having unit diagonal elements is called the Cront LU decomposition.

Q: Solve the following set of equations using LU decomposition method

$3_{x1}+2_{x2}+x_3=10$

$2x_1+3_{x2}+2_{x3}=14$

$x_1+2_{x2}+3_{x3}=14$

Making the diagonal elements of L matrix as unit i.e, using Do-Little L U decomposition, we have

$$L=\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \qquad U=\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$$\Rightarrow LU=\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ u_{11}l_{21} & u_{12}\ l_{21}+u_{22} & u_{13}\ l_{21}+u_{23} \\ u_{11}\ l_{31} & u_{12}l_{31}+u_{22}l_{32} & u_{13}\ l_{31}+u_{23}l_{32}+u_{33} \end{bmatrix}$$

We know that, A=LU

Here, A = $\begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 3 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ u_{11}l_{21} & u_{12}l_{21} + u_{22} & u_{13}l_{21} + u_{23} \\ u_{11}l_{31} & u_{12}l_{31} + u_{22}l_{32} & u_{13}l_{31} + u_{23}l_{32} + u_{33} \end{bmatrix}$$

Thus,

$U_{11}=3$ Also, $u_{11} l_{21} = 2$ $u_{11} l_{31} = 1$

$3*l_{21} = 2$ $3*l_{31} = 1$

$l_{21} = \frac{2}{3}$ $l_{31} = \frac{1}{3}$

$U_{12} = 2$ Also, $u_{12} l_{21}+u_{22} = 3$ $u_{12} l_{31}+u_{22} l_{32} = 2$

$(2)\left(\frac{2}{3}\right)+ u_{22} = 3$ $(2)\left(\frac{1}{3}\right)+\left(\frac{5}{3}\right)l_{32} = 2$

$\frac{4}{3} + u_{22} = 3$ $\frac{2}{3} + \frac{5}{3} l_{32} = 2$

$u_{22} = 3-\frac{4}{3}$ $2 + 5 l_{32} = 6$

$5 l_{32} = 4$

$u_{22} = \frac{5}{3}$ $l_{32} = \frac{4}{5}$

$U_{13} = 1$ $u_{13} l_{21} + u_{23} = 2$ $u_{13} l_{31} + u_{23} l_{32} + u_{33} = 3$

$(1)\left(\frac{2}{3}\right) + u_{23} = 2$ $(1)\left(\frac{1}{3}\right) + \left(\frac{4}{3}\right)\left(\frac{4}{5}\right) + u_{33} = 3$

$\frac{2}{3} + u_{23} = 2$ $\frac{1}{3} + \frac{16}{15} + u_{33} = 3$

$u_{23} = \frac{4}{3}$ $u_{33} = 3- \frac{1}{3} - \frac{16}{15}$

$u_{33} = \frac{8}{5}$

Thus,

$$L = \begin{matrix} 1 & 0 & 0 \\ \frac{2}{3} & 1 & 0 \\ \frac{1}{3} & \frac{4}{5} & 1 \end{matrix} \quad \text{and} \quad U = \begin{bmatrix} 3 & 2 & 1 \\ 0 & \frac{5}{3} & \frac{4}{3} \\ 0 & 0 & \frac{8}{5} \end{bmatrix}$$

Now,

Step1: $Lz = b$

Here, $b = \begin{bmatrix} 10 \\ 14 \\ 14 \end{bmatrix}$

Thus,

$$\begin{bmatrix} 1 & 0 & 0 \\ \frac{2}{3} & 1 & 0 \\ \frac{1}{3} & \frac{4}{5} & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 14 \\ 14 \end{bmatrix}$$

$$=> \begin{bmatrix} z_1 \\ \frac{2}{3} z_1 + z_2 \\ \frac{1}{3} z_1 + \frac{4}{5} z_2 + z_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 14 \\ 14 \end{bmatrix}$$

Thus,

$z_1 = 10$

Also, $\frac{2}{3} z_1 + z_2 = 14$ $\qquad$ $\frac{1}{3} z_1 + \frac{4}{5} z_2 + z_3 = 14$

$\frac{2}{3}(10) + z_2 = 14$ $\qquad$ $\frac{1}{3}(10) + \frac{4}{5}(\frac{22}{3}) + z_3 = 14$

$z_2 = \frac{22}{3}$ $\qquad$ $z_3 = \frac{24}{5}$

Hence,

$z_1 = 10$, $z_2 = \frac{22}{3}$ and $z_3 = \frac{24}{5}$

Step 2:

$Ux = Z$

$$\begin{bmatrix} 3 & 2 & 1 \\ 0 & \frac{5}{3} & \frac{4}{3} \\ 0 & 0 & \frac{8}{5} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 10 \\ \frac{22}{3} \\ \frac{24}{5} \end{bmatrix}$$

$$\begin{bmatrix} 3x_1 + & 2x_2 + & x_3 \\ 0 + & \frac{5}{3}x_2 + & \frac{4}{3}x_3 \\ 0 + & 0 + & \frac{8}{5}x_3 \end{bmatrix} = \begin{bmatrix} 10 \\ \frac{22}{3} \\ \frac{24}{5} \end{bmatrix}$$

Thus, $\frac{8}{5} x_3 = \frac{24}{5}$

$x_3 = 3$

Also, $\frac{5}{3} x_2 + \frac{4}{3} x_3 = \frac{22}{3}$

$\frac{5}{3} x_2 + \frac{4}{3} (3) = \frac{22}{3}$

=> $x_2 = 2$

And, $3x_1 + 2x_2 + x_2 = 10$

$3x_1 + 2(2) + x_3 = 10$

$3x_1 + 4 + 3 = 10$

=> $x_1 = 1$

Q: Solve the following equations using Crout's method?

$2x_1 + 8x_2 + 2x_3 = 14$

$x_1 + 6x_2 - x_3 = 13$

$2x_1 - x_2 + 2x_3 = 5$

L = $\begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}$ U = $\begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$

In Crout's method, the diagnol elements of U matrix are unity.

Now,

LU = $\begin{bmatrix} l_{11} & l_{11}u_{12} & l_{11}u_1 \\ l_{21} & l_{21}u_{12} + l_{22} & l_{21}u_{13} + l_{22}u_{23} \\ l_{31} & l_{31}u_{12} + l_{32} & l_{31}u_{13} + l_{32}u_{23} + l_{33} \end{bmatrix}$

We know that,

A = LU

Here,

A = $\begin{bmatrix} 2 & 8 & 2 \\ 1 & 6 & -1 \\ 2 & -1 & 2 \end{bmatrix}$

=> $\begin{bmatrix} 2 & 8 & 2 \\ 1 & 6 & -1 \\ 2 & -1 & 2 \end{bmatrix}$ = $\begin{bmatrix} l_{11} & l_{11}u_{12} & l_{11}u_{13} \\ l_{21} & l_{21}u_{12} + l_{22} & l_{21}u_{13} + l_{22}u_{23} \\ l_{31} & l_{31} + u_{12} + l_{32} & l_{31}u_{13} + l_{32}u_{23} + l_{33} \end{bmatrix}$

Thus,

$l_{11} = 2$, $l_{21} = 1$, $l_{31} = 2$

Also,

$l_{11}\, u_{12} = 8$

(2) $U_{12} = 8$

$U_{12} = 4$

$l_{21}\, u_{12} + l_{22} = 6$

(1) (4) + $l_{22} = 6$

$l_{22} = 2$

$l_{31}\, u_{12} + l_{32} = -1$

(2) (4) + $l_{32} = -1$

$l_{32} = -9$

$l_{11}\, u_{13} = 2$

(2) $u_{13} = 2$

$u_{13} = 1$

$l_{21}\, u_{13} + l_{22}\, u_{23} = -1$

(1) (1) + (2) $u_{23} = -1$

1 + (2) $u_{23} = -1$

$u_{23} = -1$

$l_{31}\, u_{13} + l_{32}\, u_{23} + l_{33} = 2$

(2) (1) + (-9) (-1) + $l_{33} = 2$

2+ 9 + $l_{33} = 2$

$l_{33} = -9$

Thus,

L = $\begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 2 & -9 & -9 \end{bmatrix}$ U = $\begin{bmatrix} 1 & 4 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$

Step 1:

$$Lz = b$$

$$\begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 2 & -9 & -9 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 14 \\ 13 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} 2z_1 \\ z_1 + 2z_2 \\ 2z_1 - 9z_2 - 9z_3 \end{bmatrix} = \begin{bmatrix} 14 \\ 13 \\ 5 \end{bmatrix}$$

Using forward substitution, we have

$2z_1 = 14$ $\quad$ $z_1 + 2z_2 = 13$

$z_1 = 7$ $\quad$ $7 + 2z_2 = 13$

$2z_2 = 6$

$z_2 = 3$

$2z_1 – 9z_2 – 9z_3 = 5$

$2(7) – 9(3) – 9z_3 = 5$

$14 – 27\ 9z_3 = 5$

$z_3 = -2$

Step 2:

$$Ux = z$$

$$\begin{bmatrix} 1 & 4 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \\ -2 \end{bmatrix}$$

$$\begin{bmatrix} x_1 + 4x_2 + x_3 \\ x_2 - x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \\ -2 \end{bmatrix}$$

Using backward substitution, we have.

$x_3 = -2$

Also,

$x_2 - x_3 = 3$ $\quad\quad$ $x_1 + 4x_2 + x_3 = 7$

$x_2 - (-2) = 3$ $\quad\quad$ $x_1 + 4(1) + (-2) = 7$

$x_2 = 1$ $\quad\quad$ $x_1 = 5$

Thus, the solution is

$x_1 = 5$, $x_2 = 1$ and $x_3 = -2$