NUMERICAL AND STATISTICAL COMPUTING (MCA-202-CR)

Autumn Session

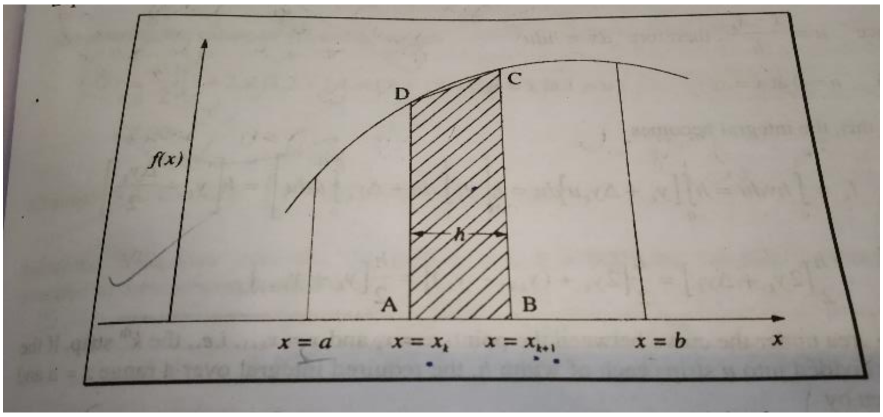
UNIT 4

Numerical Integration is the process of computing the value of a definite integral from a set of numerical values of the function referred to as integrand.

TRAPEZOIDAL RULE

The trapezoidal rule approximates the area under a curve by connecting successive points on the curve to form trapezoids of uniform width, and then summing the area under these trapezoids to obtain the approximate area under the curve.

Consider the function f(x), whose graph between x=a and x=b is shown below. An approximation to area under the curve is obtained by dividing the integral [a,b] into n strips of width h each, and approximating the area of each strip by that of a trapezoid as shown by the shaded area.



For the derivation of the formula for the trapezoid rule, we assume that the function f(x) is given in the following form.

x	x_1	x_1+h	x_1+2h	………	x_1+nh
y=f(x)	y_1	y_2	y_3	………	y_{n+1}

We consider the trapezoid formed by connecting the points (x_k,y_k) and (x_{k+1},y_{k+1}). Further, consider the first two terms of the Newton's forward difference interpolating polynomial to represent the straight line function f(x) as shown below:

$y=f(x)=y_k+\Delta y_k u$

where $\Delta y_k= y_{k+1}-y_k$ $u=(x-x_k)/h$ $h= x_{k+1}-x_k$

Then the area of the k^{th} strip is given by evaluating the following integral:

$I_k=\int_{x_k}^{x_{k+1}} y\,dx$

Since $u=(x-x_k)/h$, therefore $dx=hdu$

And $u=0$ at $x=x_k$ $u=1$ at $x=x_{k+1}$

With this, the integral becomes

$I_k= \int_0^1 hydu$ $=h\int_0^1 [y_k + \Delta y_k u]\, du$ $=h[y_k \int_0^1 du + \Delta\, y_k \int_0^1 udu]$ $=h[y_k+ \frac{\Delta\, yk}{2}]$

$=\frac{h}{2}[2y_k+\Delta\, y_k]$

$=\frac{h}{2}\,[2y_k+ (y_{k+1}-y_k)]$

$=\frac{h}{2}\,[y_k+ y_{k+1}]$

This is the area under the curve between the points $x=x_k$ and $x=x_{k+1}$, i.e., the k^{th} strip . if the function is divided into n strips each of width h, the required integral over a range x=a and x=b is given by

$I= \sum_{k=1}^{n} I_k$

$= I_1+I_2+I_3+\ldots..+I_n$

$=\frac{h}{2}\,[y_1+y_2] + \frac{h}{2}\,[y_2+y_3] +\frac{h}{2}\,[y_3+y_4]+\ldots.+ \frac{h}{2}\,[y_n+y_{n+1}]$

$I= \frac{h}{2}\,[y_1+2y_2+2y_3+2y_4+..+2y_n+y_{n+1}]$

This is the formula for the trapezoid rule.

EXAMPLE: The function f(x) is given as follows:

x	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
y	1	1.2	1.4	1.6	1.8	2.0	2.2	2.4	2.6	2.8	3.0

Calculate the integral of f(x) between x=0 and x=1.0

Sol: Given h=0.1 and n=10

Therefore,

$I= \frac{h}{2}\,[y_1+2y_2+2y_3+2y_4+..+2y_n+y_{n+1}]$ becomes

$I= \frac{h}{2}\,[y_1+2(y_2+y_3+y_4+ y_5+y_6+y_7 +y_8+y_9+y_{10})+y_{11}]$

Substituting the values

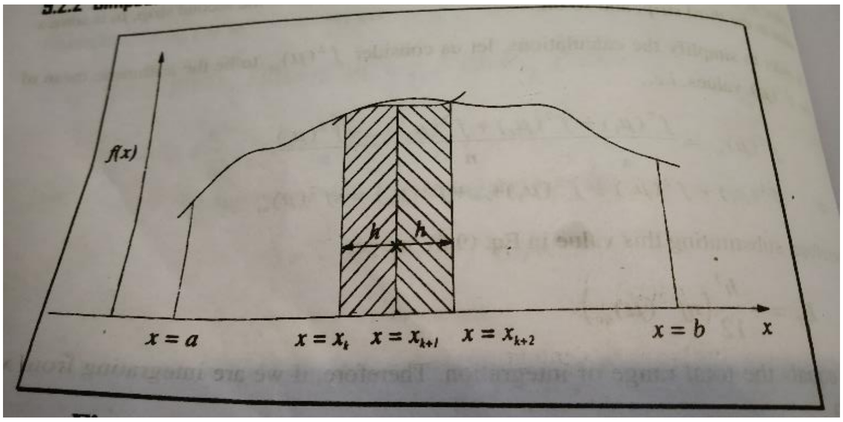
$I= \frac{0.1}{2}\,[1+2(1.2+1.4+1.6+ 1.8+2.0+2.2+2.4+2.6+2.8)+3.0]$

=2.00

SIMPSON'S 1/3rd RULE

Simpson's 1/3rd rule gives more accurate approximation of the integral value since it connects three points on the curve by second order parabolas, and then sums the areas under the parabolas to obtain the approximate area under the curve.

Consider the function f(x), whose graph between x=a and x=b is shown below. An approximation to the area under the curve is obtained by dividing it into n strips of width h each, and approximating the area of two strips by that of the area under the parabolas as shown by the shaded area.



For the derivation of the formula for the Simpson's 1/3rd rule, we assume that the function f(x) is given in the following form.

x	x_1	x_1+h	x_1+2h	………	x_1+nh
y=f(x)	y_1	y_2	y_3	………	y_{n+1}

We consider that the parabola passes through the points (x_k,y_k) , (x_{k+1},y_{k+1}) and (x_{k+2},y_{k+2}). Further consider the first three terms of the Newton's forward difference interpolating polynomial to represent parabolic function f(x) as shown:

$$y=f(x)=y_k+\Delta y_k u+\frac{\Delta^2 y^k}{2!}u(u-1)$$

where $\Delta y_k= y_{k+1}-y_k$, $\Delta^2 y_k= y_{k+2}-2y_{k+1}+y_k$, $u=(x-x_k)/h$, $h= x_{k+1}-x_k$

The area under the parabola is given by evaluating the following integral:

$$I_k= \int_{x_k}^{x_{k+2}} y\,dx$$

Since $u=(x-x_k)/h$, therefore $dx=hdu$

And $u=0$ at $x=x_k$ $\quad\quad$ $u=2$ at $x=x_{k+2}$

With this, the integral becomes

$I_k= \int_0^2 hydu$ $=h\int_0^2 \left[y_k + \Delta y_k u + \frac{\Delta^2 y_k}{2!} u(u-1)\right] du$ $=h[y_k \int_0^2 du + \Delta y_k \int_0^2 udu + (\frac{\Delta^2 y_k}{2!}) \int_0^2 u(u-1)du]$

$=h[2y_k+2\Delta y_k+(\frac{\Delta^2 y_k}{3})]$

$=\frac{h}{3}[6y_k+6\Delta y_k+\Delta^2 y_k]$

$= \frac{h}{3} [6y_k+6(y_{k+1}-y_k)+(y_{k+2}-2 y_{k+1}+ y_k)]$

$=\frac{h}{3} [y_k+4y_{k+1}+y_{k+2}]$

This is the area under the curve between the points $x=x_k$ and $x=x_{k+2}$. This covers two intervals of width h each. If we use this process repetitively n/2 times, we can get the area under the curve for n intervals. Thus, if the range of integration is divided into n intervals, each of width h, the required integral over a range x=a and x=b is given by:

$I= \sum_{k=1}^{n/2} I_{2k-1}$

$= I_1+I_3+I_5+\ldots+I_{n-1}$

$=\frac{h}{3} [y_1+4y_2+y_3] + \frac{h}{3} [y_3+4y_4+y_5] +\frac{h}{3} [y_5+4y_6+y_7]+\ldots+ \frac{h}{3} [y_{n-1}+4y_n+y_{n+1}]$

$I= \frac{h}{3} [y_1+4y_2+2y_3+4y_4+..+2y_{n-1}+4y_n+y_{n+1}]$

This is the formula for the Simpson's 1/3rd rule.

EXAMPLE: Evaluate $\int_1^2 e^{\frac{-1}{2}x}dx$ using four intervals.

Sol: With four intervals, the interval size h=0.25 the function is evaluated as (rounded to three decimal digits)

x	1	1.25	1.50	1.75	2.0
f(x)	0.607	0.535	0.472	0.417	0.368

Substituting these values we get,

$I=\frac{h}{3} [y_1+4y_2+2y_3+4y_4+y_5]$

=0.25/3[0.607+4*0.535+2*0.472+4*0.417+0.368]

=0.477

SIMPSON'S 3/8th RULE

Assume that the tabulated function can be approximated by a third order polynomial. In this case, we can use the first four terms of the Newton's forward difference interpolation polynomial to represent f(x) as shown below:

$$y=f(x)=y_k+\Delta y_k u+\frac{\Delta^2 y_k}{2!}u(u-1)+\frac{\Delta^3 y_k}{3!}u(u-1)(u-2)$$

where Δy_k, $\Delta^2 y_k$, $\Delta^3 y_k$ are forward differences at $x=x_k$, and

$u=\frac{x-x_k}{h}$, $h=x_{k+1}-x_k$

Since f(x) is a third order polynomial, we assume that it passes through the points (x_k,y_k), (x_{k+1},y_{k+1}), (x_{k+2},y_{k+2}) and (x_{k+3},y_{k+3}), then

$$I_k=\int_{x_k}^{x_{k+3}} y\,dx$$

Since $u=\frac{x-x_k}{h}$, therefore $dx=hdu$ and $u=0$ at $x=x_k$, $u=3$ at $x=x_{k+3}$

With this, the integral becomes

$$I_k=\int_0^3 hy\,du$$

After simplification, we get:

$$I_k=\frac{3h}{8}[y_k+3y_{k+1}+3y_{k+2}+y_{k+3}]$$

This is the area under the curve between the points $x=x_k$ and $x=x_{k+3}$. This covers three intervals of width h each. If we use this process repetitively n/3 times, we can get the area under the curve for n intervals.

Thus, if the range of integration is divided into n intervals, each of width h, the required integral over a range x=a and x=b is given by:

$$I=\sum_{k=1}^{n/3} I_{3k-2}=I_1+I_4+I_7+..+I_{n-2}$$

$$=\frac{3h}{8}[y_1+3y_2+3y_3+y_4]+\frac{3h}{8}[y_4+3y_5+3y_6+y_7]$$

$$+\frac{3h}{8}[y_7+3y_8+3y_9+y_{10}]+..+\frac{3h}{8}[y_{n-2}+3y_{n-1}+3y_n+y_{n+1}]$$

$$I=\frac{3h}{8}[y_1+3y_2+3y_3+2y_4+3y_5+3y_6+2y_7+..+2y_{n-2}+3y_{n-1}+3y_n+y_{n+1}]$$

This is the formula for Simpson's 3/8th Rule.

EXAMPLE: The function f(x) is given as follows:

x	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
y	1.001	1.008	1.027	1.064	1.125	1.216	1.343	1.512	1.729	2.0

Compute the integral of f(x) between x=0.1 and x=1.0.

Sol: Given h=0.1 and n=9

Therefore,

I=$\frac{3h}{8}[y_1 + 3y_2 + 3y_3 + 2y_4 + 3y_5 + 3y_6 + 2y_7 + 3y_8 + 3y_9 + y_{10}]$

Substituting the values,

I=$\frac{3*0.1}{8}$[1.001+3*1.008+3*1.027+2*1.064+3*1.125+3*1.216+2*1.343+3*1.512+3*1.729+2.0]

=1.150

Order of Differential Equation: The order of the differential equation is the order of the highest order derivative in the differential equation. e.g.,

$m\frac{d^2y}{dx^2}+ c\frac{dy}{dx} + k\,y= 0$ has order 2.

Degree of Differential Equation: The degree of the differential equation is the power of the highest order derivative in the differential equation. e.g.,

$[\frac{d^2y}{dx^2}]^2 + c\frac{dy}{dx} +k\,y =0$ has degree 2.

Solution of the Differential Equation: The solution of a differential equation is a 2D curve g(x,y) in the x,y plane where slope at every point (x,y) in the specified region is given by:

$$\frac{dy}{dx} = f(x,y)$$

or, $$y' = f(x, y)$$

Euler's Method

The Euler's method gives a technique of developing a piece wise linear approximation to the solution. Consider the first order differential equation.

$$\frac{dy}{dx} = f(x,y)$$

with initial condition $y=y_1$ for $x=x_1$.

As per the mean value theorem, if a function is continuous and differentiable between the points, (x_1, y_1) and (x_2, y_2), then the slope of the line joining these points is equal to the derivative of the function at least at one other point (c, d) between these two points i.e.,

$$y'(c) = \frac{y(x_2)-y(x_1)}{x_2-x_1} \qquad (1)$$

substituting $c=x_1$ and $h=x_2-x_1$. The above equation becomes:

$$y'(x_1) = \frac{y(x_2)-y(x_1)}{h}$$

$y'(x_1)h = y(x_2)- y(x_1)$ or,

$$y(x_2)- y(x_1) = y'(x_1)h \qquad (2)$$

Now, we know that $y'(x_1)=f(x_1,y_1)$

Therefore, substituting values of $y'(x_1)$ in equation (2), we have

$$y(x_2)- y(x_1)= h.f(x_1,y_1)$$

$$y(x_2)= h.f(x_1,y_1)+ y(x_1)$$

$$y(x_2)= y(x_1)+ h.f(x_1,y_1)$$

$$y_2= y_1+ h.f(x_1,y_1)$$

Similarly, taking (x_2,y_2) as the starting point, we have

$$y_3= y_2+ h.f(x_2,y_2)$$

Similarly, $y_4= y_3+h.f(x_3,y_3)$

Thus, $y_n= y_{n-1}+ h.f(x_{n-1},y_{n-1})$

And, $y_{n+1}= y_n+ h.f(x_n,y_n)$

This equation is known as the Euler's Formula.

EXAMPLE1: Given $\frac{dy}{dx} = xy$, with y(1)=5. Find the solution correct to three decimal positions in the interval (1,1.5) with step size h=0.1.

Sol: The Euler's formula is

$$y_{n+1}= y_n+ h.f(x_n,y_n)$$

Given, $x_1=1$, $y_1=5$ and h=0.1.

Thus, the Euler's formula can be written as:

$$y_{n+1}= y_n+ 0.1\ (xy)$$

ITERATION1:

y(1.1)= y(1)+ 0.1 (xy)

= 5+0.1(1)(5)

= 5+0.5=5.5

ITERATION2:

y(1.2)= y(1.1)+ 0.1 (xy)

=5.5+0.1(1.1)(5.5)

=5.5+0.605=6.105

ITERATION3:

y(1.3)= y(1.2)+ 0.1 (xy)

=6.105+0.1(1.2)(6.105)

=6.105+0.7326=6.838

ITERATION4:

y(1.4)= y(1.3)+ 0.1 (xy)

=6.838+0.1(1.3)(6.838)

=6.838+0.88894=7.727

ITERATION5:

y(1.5)= y(1.4)+ 0.1 (xy)

=7.727+0.1(1.4)(7.727)

=7.727+1.08178=8.809

The complete solution of the problem is given as:

x	1	1.1	1.2	1.3	1.4	1.5
y	5	5.5	6.105	6.838	7.727	8.809

TAYLOR SERIES METHOD

Given $\frac{dy}{dx}$ = f(x,y) with an initial condition: $y=y_1$ at $x=x_1$.

If the solution curve is expanded in Taylor Series around $x=x_1$, we get

$y(x) = y(x_1) + (x-x_1)y^1(x_1) + \frac{(x-x_1)^2}{2!}.\ y^2(x_1) + \ldots.$

where $y^1(x_1)$, $y^2(x_1)$,…. are first and second order derivatives, and so on.

Then at $x=x_1+h$, the above equation becomes:

$y(x_1+h) = y(x_1) + h\ y^1(x_1) + (\frac{h^2}{2!}).\ y^2(x_1) + \ldots$

Now, $y^1(x_1) = f(x,y)$ and y is a function of x, therefore

$y^2(x) = \frac{d[y^1(x)]}{dx} = \frac{d}{dx}[f(x,y)] = \frac{\partial}{\partial x} f(x,y) + \frac{\partial}{\partial y} f(x,y) \frac{dy}{dx}$

$= \frac{\partial}{\partial x} f(x,y) + \frac{\partial}{\partial y} f(x,y) * f(x,y)$

If we take, $f_x(x,y) = \frac{\partial}{\partial x} f(x,y)$ and $f_y(x,y) = \frac{\partial}{\partial y} f(x,y)$

Then $y^2(x) = f_x(x,y) + f_y(x,y) * f(x,y)$

Thus $y(x_1+h) = y(x_1) + hf(x_1,y_1) + (\frac{h^2}{2!}).[\ f_x(x_1,y_1) + f_y(x_1,y_1) * f(x_1,y_1)] + \ldots$

$y_2 = y(x_1+h) = y_1 + hf(x_1,y_1) + (\frac{h^2}{2!}).[\ f_x(x_1,y_1) + f_y(x_1,y_1) * f(x_1,y_1)] + \ldots$

Similarly, taking (x_2,y_2) as the starting point, we get

$y_3 = y_2 + hf(x_2,y_2) + (\frac{h^2}{2!}).[\ f_x(x_2,y_2) + f_y(x_2,y_2) * f(x_2,y_2)] + \ldots$

In general,

$y_{i+1} = y_i + hf(x_i,y_i) + (\frac{h^2}{2!}).[\ f_x(x_i,y_i) + f_y(x_i,y_i) * f(x_i,y_i)] + \ldots$

PREDICTOR CORRECTOR METHODS

These are the multi-step methods that use the past information of the curve to extrapolate the solution curve. Some of the predictor corrector methods use the information about the solution curve at two previous points, some use at three points, still others use at more. The only problem with predictor corrector methods, excluding Modified Euler’s method, is that they are not self starting, i.e., the values at first few points are computed using some other methods and then from there on these methods can take on.

Some of the predictor corrector methods are:

1. Modified Euler’s Method
2. Adams-Bashforth Method
3. Milne’s Method

Modified Euler's Method

It is seen in the basic Euler's method that we use the slope at the starting point of the solution curve to determine the next point of the solution curve. This technique will work correctly only if the function is linear. The alternate approach is to use the average slope within the interval. This can be approximated by the mean of the slopes at both end points of the interval.

Suppose we use interval bounded by the point $x=x_i$ and $x=x_{i+1}$, then the average slope is:

$$(y^1(x_i)+ y^1(x_{i+1}))/2$$

Then we have,

$$y_{i+1}= y_i+ \frac{h}{2} [y^1(x_i)+ y^1(x_{i+1})]$$

$$y_{i+1}= y_i+ \frac{h}{2} [f(x_i,y_i)+ f(x_{i+1}, y_{i+1})]$$

This is an improved estimate for y_{i+1} at x_{i+1}. But we are unable to apply the above equation directly since we cannot evaluate $f(x_{i+1}, y_{i+1})$ as the value of y_{i+1} is unknown. This method works by estimating or predicting the value of y_{i+1} by basic Euler's method. Then it uses this value to compute $f(x_{i+1}, y_{i+1})$ giving an improved estimate (a corrected value) for y_{i+1}. Thus, the value of y_{i+1} is predicted using the equation.

$$y^p_{i+1} = y_i +h\, f(x_i,y_i)$$

This equation is known as the predictor formula.

Using this predicted value of y i.e. y^p_{i+1}, a more accurate value of y_{i+1} is computed using the equation

$$y^c_{i+1} = y_i + \frac{h}{2} [f(x_i,y_i)+ f(x_{i+1}, y^p_{i+1})]$$

This equation is known as the corrector formula.

Thus, these two equations i.e. the predictor formula and the corrector formula constitute the Euler's predictor corrector method.

EXAMPLE:

Given $\frac{dy}{dx}$=xy with y(1) =5. Find the solution in the interval [1,1.5] using step size h=0.1.

Sol: The formula for predictor corrector method is:

$$y^p_{i+1} = y_i +h\, f(x_i,y_i)$$

$y^{c}_{i+1} = y_i + \frac{h}{2} [f(x_i,y_i)+ f(x_{i+1}, y^{p}_{i+1})]$

ITERATION1:

Starting with initial conditions, we first predict the value of y_2 using the predictor formula.

$y^{p}_{2} = y_1 + h f(x_1,y_1)=5+0.1*f(1,5)=5+0.1*1*5=5.5$

Using the predicted value of y_2, we make the correction on it using the corrector formula.

$y^{c}_{2} = y_1 + \frac{h}{2} [f(x_1,y_1)+ f(x_2, y^{p}_2)]$

=5+(0.1/2)[f(1,5)+ f(1.1,5.5)]= 5+(0.1/2)[1*5+1.1*5.5]=5.553.

Thus, the second solution point is obtained as (x_2,y_2)=(1.1, 5.553)

ITERATION2:

Starting with second solution point (x_2,y_2)=(1.1, 5.553), we first predict the value of y_3 using the predictor formula.

$y^{p}_{3} = y_2 + h f(x_2,y_2)=5.553+0.1*f(1.1,5.553)=5.553+0.1*1.1*5.553=6.164.$

Using the predicted value of y_3, we make the correction on it using the corrector formula.

$y^{c}_{3} = y_2 + \frac{h}{2} [f(x_2,y_2)+ f(x_3, y^{p}_3)]$

=5.553+(0.1/2)[f(1.1,5.553)+ f(1.2,6.164)]= 5.553+(0.1/2)[1.1*5.553+1.2*6.164]=6.231.

Thus, the third solution point is obtained as (x_3,y_3)=(1.2, 6.231)

ITERATION3:

Starting with third solution point (x_3,y_3)=(1.2, 6.231), we first predict the value of y_4 using the predictor formula.

$y^{p}_{4} = y_3 + h f(x_3,y_3)=6.231+0.1*f(1.2,6.231)=6.231+0.1*1.2*6.231=6.979.$

Using the predicted value of y_4, we make the correction on it using the corrector formula.

$y^{c}_{4} = y_3 + \frac{h}{2} [f(x_3,y_3)+ f(x_4, y^{p}_4)]$

=6.231+(0.1/2)[f(1.2,6.231)+ f(1.3,6.979)]= 6.231+(0.1/2)[1.2*6.231+1.3*6.979]=7.059.

Thus, the fourth solution point is obtained as (x_4,y_4)=(1.3, 7.059)

ITERATION4:

Starting with fourth solution point (x_4,y_4)=(1.3, 7.059), we first predict the value of y_5 using the predictor formula.

$y^p_5 = y_4 + h\, f(x_4,y_4)$=7.059+0.1*f(1.3,7.059)=7.059+0.1*1.3*7.059=7.977.

Using the predicted value of y_5, we make the correction on it using the corrector formula.

$$y^c_5 = y_4 + \frac{h}{2}[f(x_4,y_4)+ f(x_5, y^p_5)]$$

=7.059+(0.1/2)[f(1.3,7.059)+ f(1.4,7.977)]= 7.059+(0.1/2)[1.3*7.059+1.4*7.977]=8.076.

Thus, the fifth solution point is obtained as (x_5,y_5)=(1.4, 8.076)

ITERATION5:

Starting with fifth solution point (x_5,y_5)=(1.4, 8.076), we first predict the value of y_6 using the predictor formula.

$y^p_6 = y_5 + h\, f(x_5,y_5)$=8.076+0.1*f(1.4,8.076)=8.076+0.1*1.4*8.076=9.207.

Using the predicted value of y_6, we make the correction on it using the corrector formula.

$$y^c_6 = y_5 + \frac{h}{2}[f(x_5,y_5)+ f(x_6, y^p_6)]$$

=8.076+(0.1/2)[f(1.4,8.076)+ f(1.5,9.207)]= 8.076+(0.1/2)[1.4*8.076+1.5*9.207]=9.332.

Thus, the sixth and final solution point is obtained as (x_6,y_6)=(1.5, 9.332).

The complete solution of the given differential equations is given as:

i	1	2	3	4	5	6
x_i	1.0	1.1	1.2	1.3	1.4	1.5
y_i	5.0	5.553	6.231	7.059	8.076	9.332

RUNGE-KUTTA METHODS:

The Runge-Kutta methods are actually a family of methods, of which the second order and fourth order methods are widely used. Much greater accuracy can be obtained by using Runge-Kutta methods. These methods are equivalent of approximating the exact solution by matching the first n terms of the Taylor Series Method.

RUNGE-KUTTA SECOND ORDER METHODS

The Runge-Kutta second order methods are actually a family of methods, each of that matches the Taylor Series method up to the second-degree terms in h, where h is the step size. In these methods the interval[x_1,x_f] is divided into subintervals and a weighted average of derivatives (slopes) at these intervals is used to determine the value of the dependent variable.

One advantage of these methods is that they, like Euler's method, are single step methods i.e. in order to evaluate y_{i+1}, we need information only at the preceeding point (x_i,y_i).

Consider the following differential equation

$$\frac{dy}{dx} = f(x, y)$$

With an initial condition:

$y=y_1$ at $x=x_1$

At the starting point, compute the slope of the curve as $f(x_1,y_1)$. Let it be s_1. Now compute the slope of the curve at point (x_2,y_1+s_1h) as $f(x_2,y_1+s_1h)$, where $x_2=x_1+h$. Let this new slope be s_2. Find the average of these two slopes, and then find the value of the dependent variable y from the following equation

$y_2= y_1+hs$

where $s=(s_1+s_2)/2$, $s_1= f(x_1,y_1)$ and $s_2= (x_1+h, y_1+s_1h)$

Hence, starting from point $f(x_1,y_1)$, we obtained the second point (x_2,y_2). Similarly, starting from second point, we can obtain the third point. And this process is repeated till we find the solution in the desired interval.

In general, the value of y for the $(i+1)^{th}$ point on the solution curveis obtained from the i^{th} solution point using the formula.

$$y_{i+1}=y_i +hs$$

where $s=(s_i+s_{i+1})/2$, $s_i=f(x_i.y_i)$, and $s_{i+1}=(x_i+h, y_i+s_ih)$

This formula for the Runge-Kutta second order method is also known as Heun's method.

EXAMPLE:

Given $\frac{dy}{dx}$=xy with y(1) =5. Find the solution correct to three decimal position in the interval [1,1.5] using step size h=0.1.

Sol:

The formula for second order Runge-Kutta method is:

$$y_{i+1}=y_i +hs$$

where $s=(s_1+s_2)/2$, $s_1=f(x_1,y_1)$, and $s_2=(x_1+h, y_1+s_1h)$.

In our example, $f(x,y)=xy$ $x_1=1$ $y_1=5$ $h=0.1$

Runge-Kutta second order formula can be written as:

$$y_{i+1}=y_i +0.1*s$$

ITERATION1:

y_2=y_1+0.1*s

s_1=f(x_1,y_1)=f(1,5)=1*5=5

s_2=f(x_1+0.1,y_1+0.1*s1)=f(1.1,5+0.1*5)=f(1.1,5.5)=1.1*5.5=6.05

s=(5+6.05)/2=5.525

Thus y_2=5+0.1*5.525=5.553

Thus, the second solution point is obtained as (x_2,y_2)=(1.1,5.553)

ITERATION2:

y_3=y_2+0.1*s

s_1=f(x_2,y_2)=f(1.1,5.553)=1.1*5.553=6.108

s_2=f(x_2+0.1,y_2+0.1*s_1)=f(1.2,5.553+0.1*6.108)=f(1.2,6.164)=1.2*6.164=7.397

s=(6.108+7.397)/2=6.752

Thus y_3=5.553+0.1*6.752=6.228

Thus, the third solution point is obtained as (x_3,y_3)=(1.2,6.228)

ITERATION3:

y_4=y_3+0.1*s

s_1=f(x_3,y_3)=f(1.2,6.228)=1.2*6.228=7.474

s_2=f(x_3+0.1,y_3+0.1*s_1)=f(1.3,6.228+0.1*7.474)=f(1.3,6.975)=1.3*6.975=9.068

s=(7.474+9.068)/2=8.271

Thus y_4=6.228+0.1*8.271=7.055

Thus, the fourth solution point is obtained as (x_4,y_4)=(1.3,7.055)

ITERATION4:

y_5=y_4+0.1*s

s_1=f(x_4,y_4)=f(1.3,7.055)=1.3*7.055=9.172

s_2=f(x_4+0.1,y_4+0.1*s_1)=f(1.4,7.055+0.1*9.172)=f(1.4,7.972)=1.4*7.972=11.161

s=(9.172+11.161)/2=10.166

Thus y_5=7.055+0.1*10.166=8.072

Thus, the fifth solution point is obtained as (x_5,y_5)=(1.4,8.072)

ITERATION5:

$y_6=y_5+0.1*s$

$s_1=f(x_5,y_5)=f(1.4,8.072)=1.4*8.072=11.301$

$s_2=f(x_5+0.1,y_5+0.1*s_1)=f(1.5,8.072+0.1*11.301)=f(1.5,9.202)=1.5*9.202=13.803$

s=(11.301+13.803)/2=12.552

Thus $y_6=8.072+0.1*12.552=9.327$

Thus, the sixth and final solution point is obtained as $(x_6,y_6)=(1.5,9.327)$

The complete solution of the given differential equation is given as:

i	1	2	3	4	5	6
x_i	1.0	1.1	1.2	1.3	1.4	1.5
y_i	5.0	5.525	6.228	7.055	8.072	9.327

RUNGE-KUTTA FOURTH ORDER METHODS:

The error in the second order Runge-Kutta methods is $O(h^3)$ per step. However, if more precision is required, then we can use the fourth order Runge-Kutta methods in which the error is $O(h^5)$ per step.

In Runge-Kutta fourth order methods, the slope at four points including the starting point is computed, and then the weighted average of these slopes is computed as:

$s=1/6(s_1+2s_2+2s_3+s_4)$

where $s_1=f(x_1,y_1)$, $s_2= f(x_1+h/2,y_1+(h/2)s_1)$

$s_3= f(x_1+h/2,y_1+(h/2)s_2)$, $s_4= f(x_1+h,y_1+hs_3)$

The value of the dependent variable y is computed as:

$y_2=y_1+hs$

In the similar manner, starting from the second solution point we compute the third point. This process is repeated till the solution is found in the desired interval.

In general, the $(i+1)^{th}$ point of the solution curve is obtained from the i^{th} point using the following equation.

$y_{i+1}= y_i+hs$

where $s=1/6(s_1+2s_2+2s_3+s_4)$, $s_1=f(x_i,y_i)$, $s_2= f(x_i+h/2,y_i+(h/2)s_1)$

$s_3= f(x_i+h/2,y_i+(h/2)s_2)$, $s_4= f(x_i+h,y_i+hs_3)$

EXAMPLE:

Given $\frac{dy}{dx}$=xy with y(1) =5. Find the solution correct to three decimal position in the interval [1,1.5] using step size h=0.1.

Sol:

The formula for fourth order Runge-Kutta method is:

$$y_{i+1}=y_i+hs$$

where $s=1/6(s_1+2s_2+2s_3+s_4)$, $s_1=f(x_i,y_i)$, $s_2= f(x_i+h/2,y_i+(h/2)s_1)$

$s_3= f(x_i+h/2,y_i+(h/2)s_2)$, $s_4= f(x_i+h,y_i+hs_3)$.

In our example

f(x,y)=xy

$x_1=1$ $y_1=5$ h=0.1

The Runge-Kutta fourth order formula can be written as:

$y_{i+1}=y_i+0.1*s$

ITERATION1:

$y_2=y_1+0.1*s$

$s_1=f(x_1,y_1)=f(1,5)=1*5=5$

$s_2=f(x_1+0.05,y_1+0.05*s_1)=f(1.05,5+0.05*5)=f(1.05,5.25)$

=1.05*5.25=5.513

$s_3=f(x_1+0.05,y_1+0.05*s_2)=f(1.05,5+0.05*5.513)=f(1.05,5.276)$

=1.05*5.276=5.540

$s_4=f(x_1+0.1,y_1+0.1*s_3)=f(1.1,5+0.1*5.540)=f(1.1,5.554)$

=1.1*5.554=6.109

s=1/6[5+2*5.513+2*5.540+6.109]=5.536

Therefore, $y_2=5+0.1*5.536=5.554$

Thus, the second solution point is obtained as $(x_2,y_2)=(1.1,5.554)$

ITERATION2:

$y_3=y_2+0.1*s$

$s_1=f(x_2,y_2)=f(1.1,5.554)=1.1*5.554=6.109$

$s_2=f(x_2+0.05,y_2+0.05*s_1)=f(1.15,5.554+0.05*6.109)=f(1.15,5.859)$

$=1.15*5.859=6.738$

$s_3=f(x_2+0.05,y_2+0.05*s_2)=f(1.15,5.554+0.05*6.738)=f(1.15,5.891)$

$=1.15*5.891=6.775$

$s_4=f(x_2+0.1,y_2+0.1*s_3)=f(1.2,5.554+0.1*6.775)=f(1.2,6.232)$

$=1.2*6.232=7.478$

$s=1/6[6.109+2*6.738+2*6.775+7.478]=6.769$

Therefore, $y_3=5.554+0.1*6.769=6.231$

Thus, the third solution point is obtained as $(x_3,y_3)=(1.2,6.231)$

ITERATION3:

$y_4=y_3+0.1*s$

$s_1=f(x_3,y_3)=f(1.2,6.231)=1.2*6.231=7.477$

$s_2=f(x_3+0.05,y_3+0.05*s_1)=f(1.25,6.231+0.05*7.477)=f(1.25,6.604)$

$=1.25*6.604=8.256$

$s_3=f(x_3+0.05,y_3+0.05*s_2)=f(1.25,6.231+0.05*8.256)=f(1.25,6.644)$

$=1.25*6.644=8.305$

$s_4=f(x_3+0.1,y_3+0.1*s_3)=f(1.3,6.231+0.1*8.305)=f(1.3,7.062)$

$=1.3*7.062=9.181$

$s=1/6[7.477+2*8.256+2*8.305+9.181]=8.297$

Therefore, $y_4=6.231+0.1*8.297=7.061$

Thus, the fourth solution point is obtained as $(x_4,y_4)=(1.3,7.061)$

ITERATION4:

$y_5=y_4+0.1*s$

$s_1=f(x_4,y_4)=f(1.3,7.061)=1.3*7.061=9.179$

$s_2=f(x_4+0.05,y_4+0.05*s_1)=f(1.35,7.061+0.05*9.179)=f(1.35,7.520)$

$=1.35*7.520=10.152$

s_3=f(x_4+0.05,y_4+0.05*s_2)=f(1.35,7.061+0.05*10.152)=f(1.35,7.569)

=1.35*7.569=10.218

s_4=f(x_4+0.1,y_4+0.1*s_3)=f(1.4,7.061+0.1*10.218)=f(1.4,9.402)

=1.4*9.402=13.163

s=1/6[9.179+2*10.152+2*10.218+13.163]=10.514

Therefore, y_5=7.061+0.1*10.514=8.112

Thus, the fifth solution point is obtained as (x_5,y_5)=(1.4,8.112)

ITERATION5:

y_6=y_5+0.1*s

s_1=f(x_5,y_5)=f(1.4,8.112)=1.4*8.112=11.357

s_2=f(x_5+0.05,y_5+0.05*s_1)=f(1.45,8.112+0.05*11.357)=f(1.45,8.680)

=1.45*8.680=12.586

s_3=f(x_5+0.05,y_5+0.05*s_2)=f(1.45,8.112+0.05*12.586)=f(1.45,8.741)

=1.45*8.741=12.675

s_4=f(x_5+0.1,y_5+0.1*s_3)=f(1.5,8.112+0.1*12.675)=f(1.5,9.380)

=1.5*9.380=14.070

s=1/6[11.357+2*12.586+2*12.675+14.070]=12.658

Therefore, y_6=8.112+0.1*12.658=9.378

Thus, the sixth and final solution point is obtained as (x_6,y_6)=(1.5,9.378)

The complete solution of the given differential equations is given as:

i	1	2	3	4	5	6
x_i	1.0	1.1	1.2	1.3	1.4	1.5
y_i	5.0	5.554	6.231	7.061	8.112	9.378

EXERCISE:

1. Use the Taylor method recursively to solve the equation

$$y' = x^2+y^2,\ y(0)=0$$

for the interval (0,0.4) using two subintervals of size 0.2.

2. Comparison of Runge-Kutta and predictor and correction methods.

3. Write programmatic implementations of the above methods in C or C++.