MATHEMATICAL FORMULATION OF TRANSPORTATION PROBLEMS:

The classic transportation problem may be stated mathematically as follows:
Let a_i = quantity of product available at origin i.

b_j =quantity of product required at destination j.

c_{ij} =the cost of transporting one unit of product from origin I to destination j.

x_{ij} =the quantity transported from origin i to destination j.

Assuming that $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$, which means that the total quantity available at the origins is precisely equal to the total amount required at the destinations.
With these, the problem can be stated as an LPP:

$$\textit{Minimize} \qquad \textit{Total cost } Z = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij}$$

subject to

$$\sum_{j=1}^{n} x_{ij} = a_i \qquad \textit{for } i = 1, 2, \ldots, m$$

$$\sum_{i=1}^{m} x_{ij} = b_j \qquad \textit{for } j = 1, 2, \ldots, n$$

$$\textit{and} \qquad x_{ij} \geq 0 \qquad \textit{for all } i = 1, 2, \ldots, m \textit{ and } j = 1, 2, \ldots, n$$

The transportation model can be portrayed in a tabular form by means of a transportation tableau as shown below:

Origin (i)	Destination (j)				Supply, a_i
	1	2	…	n	
1	c_{11} x_{11}	c_{12} x_{12}	…	c_{1n} x_{1n}	a_1
2	c_{21} x_{21}	c_{22} x_{22}	…	c_{2n} x_{2n}	a_2
…	…	…	…	…	…
M	c_{m1} x_{m1}	c_{m2} x_{m2}	…	c_{mn}	a_m x_{mn}
Demand, b_j	b_1	b_2	…	b_n	$\sum a_i = \sum b_j$

The above table can be thought of as a matrix within a matrix, of the dimension mxn. One is the per unit cost matrix which represents the unit transportation costs for each of the possible transportation routes. Its elements are given by c_{ij} , indicating the cost of shipping a unit from the i^{th} origin to the j^{th} destination. Superimposed on this matrix is the matrix in which each cell contains a transportation variable, i.e. the number of units shipped from the row designated 'origin' to the column designated 'destination'. Each such variable is represented by x_{ij}, the amount shipped from i^{th} to the j^{th} destination. Right and bottom sides of the transportation table show, respectively, the amount of supplies a_i available at source i and the amount demanded b_j at each destination j. The a_i's and b_j's represent the supply and demand constraints.
The aggregate transportation cost is determined by multiplying the various x_{ij}'s with corresponding c_{ij}'s and then adding them up all. The solution to the transportation problem calls for determining values of x_{ij}'s as would yield the minimum aggregate transportation costs. When a problem is solved, some of the x_{ij}'s would assume positive values indicating utilized routes. The cells containing such values are called occupied or filled cells and each of them represents the presence of a basic variable. For the remaining cells, called empty cells, x_{ij}'s would be zero. These are the routes that are not utilized by the transportation pattern and their corresponding variables (x_{ij}'s) are regarded to be non-basic.

THE TRANSPORTATION METHOD:

This method is developed to solve transportation problems. Unlike simplex algorithm, it does not require any artificial variables to introduce for obtaining an initial solution to the problem. This method involves three main steps:

1. Obtaining an initial solution, i.e. to say make an initial assignment in such a way that a basic feasible solution is obtained.
2. Ascertaining whether it is optimal or not, by determining opportunity costs associated with the empty cells. If the solution is optimal then we stop, otherwise we proceed to step 3.
3. Revise the solution until an optimal solution is reached.

Step 1: The first step in using transportation method is to obtain a feasible solution i.e. the one that satisfies the requirements of demands and supply. Various methods can be used to obtain the initial feasible solution. These include:

- North-West Corner Rule
- Least Cost Method or Matrix minima method
- Vogels approximation method

NORTH-WEST CORNER RULE:

This rule may be stated as follows.

We start with the north-west corner of the transportation tableau that is the cell representing the first column and the first row. Corresponding to this cell are the values a_1 (i.e. supply) and b_1 (i.e. demand). According to the north-west rule we proceed as follows:

If $a_1>b_1$, then assign quantity b_1 to this cell i.e. put $x_{11}=b_1$.

If $a_1<b_1$, then assign a_1 in the cell so that $x_{11}=a_1$.

If$a_1=b_1$, then $x_{11}=a_1=b_1$.

Now, if $a_1>b_1$, then move horizontally to the next column in the first row; if $a_1<b_1$, move vertically to the same column to the next row; and if $a_1=b_1$, then move diagonally to the next column and next row.

In other words, $a_1>b_1$ means supply is greater than demand and having assigned quantity equal to demand (b_1), the remaining quantity is considered along with demand at the next destination (b_2), whereas $a_1<b_1$ means supply falls short of demand, in which case having exhausted the available supply at source 1, we consider obtaining from the next source (a_2). If supply and demand match then we consider the next source (a_2) and destination (b_2).

Once in the next cell, we again compare the supply available at the source and demand at the destination, corresponding to the cell chosen and assign lower of the two values. This process continues until the last source and destination are covered.

Example: **A firm owns facilities at seven places. It has manufacturing plants at places A,B and C with daily output of 500,300 and 200 units of an item respectively. It has warehouse at places P,Q,R and S with daily requirements of 180,150,350 and 320 units respectively. Per unit shipping charges on different routes are given below:**

To:	*P*	*Q*	*R*	*S*
From A:	12	10	12	13
From B:	7	11	8	14
From C:	6	16	11	7

The firm wants to send the output from various plants to warehouses involving minimum transportation cost. How should it route the product so as to achieve its objective?

Solution: First of all we formulate the initial transportation tableau for the above problem and then implement the north-west corner rule to obtain the initial feasible solution as depicted below:

We start with the cell AP. Row total corresponding to this cell is 500 and the column total is 180. So allocate 180 units to this cell. With the destination requirement being satisfied move horizontally to cell AQ. With the supply available in the plant A (a_1) being 320 and demand at warehouse Q (b_2) being 150 units, allocate 150 units to this cell and move to cell AR. Now the

To → / From ↓	P	Q	R	S	Supply
A	12 [180]	10 [150]	12 [170]	13	500
B	7	11	8 [180]	14 [120]	300
C	6	16	11	7 [200]	200
Demand	180	150	350	320	1000

Table 1: Initial Feasible Solution : NWC Method

supply left at a_1 is only 170 units whereas the demand b_3=350. So we assign 170 units to cell AR and move vertically down to cell BR. Now the row and columns total are a_2=300 and b_3=180. Thus we assign 180 units to cell BR and move to cell BS. With current a_2=120 and b_4=320, allocate 120 units to cell BS. Finally shift to cell CS corresponding to which a_3 and b_4 are both equal to 200. So we assign this quantity to cell CS.

This routing of units meets all the requirements of demand and supply and involves a total of 6 shipments as there are a total of 6 occupied cells.

$$Total\ cost = 12\times180 + 10\times150 + 12\times170 + 8\times180 + 14\times120 + 7\times200 = Rs\,10,220$$

Although this method of obtaining the initial feasible solution is relatively simple, it is not considered very efficient in terms of cost minimizing. This is because it takes into account only the available supply and demand requirements in making assignments and takes not account of the shipping cost involved. It thus ignores the very factor (i.e. cost) which is sought to be minimized.

LEAST COST METHOD (LCM):

This method entails making allocation at each step by selecting from the routes (i.e. combination of sources and destinations) available, the one with the minimum cost. We begin by choosing the cell with minimum cost and its corresponding supply and demand and assign the lower of the two to the cell. After this we delete the row or column or both (if a_i=b_j), whichever is satisfied by the allocation. If the row is deleted, then the column is revised by subtracting the quantity assigned. Similarly if the column is deleted, then the row total is revised. Following this step we consider the remaining routes and again choose the one with least cost, make assignment and adjust the row/column total. We continue in this manner until all the units are assigned.

If there is a tie in the minimum cost such that two or more routes have the same least cost of shipping, then conceptually either of them can be selected. However a better initial solution is obtained if the route chosen is the one where largest quantity can be assigned. Thus if there are three cells for which least cost value is equal then consider all of these one by one and determine the quantity (by reference to the demand and supply quantities given) which can be dispatched and choose the cell with the largest quantity. If there is still a tie then either of them may be selected.

Example: **A firm owns facilities at seven places. It has manufacturing plants at places A,B and C with daily output of 500,300 and 200 units of an item respectively. It has warehouse at places P,Q,R and S with daily requirements of 180,150,350 and 320 units respectively. Per unit shipping charges on different routes are given below:**

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From A:	12	10	12	13
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From C:	6	16	11	7

The firm wants to send the output from various plants to warehouses involving minimum transportation cost. How should it route the product so as to achieve its objective?

Solution: First of all we formulate the initial transportation tableau for the above problem and then implement the least cost method rule to obtain the initial feasible solution as depicted below:

We start with the cell CP which represents the lowest of all cost elements with its corresponding supply and demand being 200 and 180 units respectively. Accordingly, assign 180 to cell CP, delete the column headed P and adjust the row to 20. Of the remaining cost elements (excluding those of the deleted column), the least is 7 corresponding to cell CS. The quantity assigned to this cell is 20 being lower of demand (=320) and supply (=20). Delete the row C and adjust the demand for S at 300. The least cost element in the remaining cell is 8, for BR. With the relevant supply and demand values equal to 300 and 350 units respectively, assign 300 to cell BR. Delete row B and adjust the demand at R to 50. Now with only one supply source remaining, the amount will be transferred to the current requirements at warehouses Q , R and S: 150 to Q, 50 to R and 300 to S.

To → / From ↓	P	Q	R	S	Supply
A	12	10 [150]	12 [50]	13 [300]	500
B	7	11	8 [300]	14	300
C	6 [180]	16	11	7 [20]	~~200~~ 20
Demand	180	150	~~350~~ 50	~~320~~ 300	1000

Table2: Initial Feasible Solution : LCM Method

The transportation schedule obtained above is reproduced as shown below:

To → / From ↓	P	Q	R	S	Supply
A	12	10 [150]	12 [50]	13 [300]	500
B	7	11	8 [300]	14	300
C	6 [180]	16	11	7 [20]	200
Demand	180	150	350	320	1000

$$Total\ cost = 10\times150+12\times50+13\times300+8\times300+6\times180+7\times20 = Rs\ 9{,}620$$

VOGEL'S APPROXIMATION METHOD (VAM):

This method uses cost differentials rather than absolute costs to select the appropriate cell to make allocation. This method can be discussed as follows:

First consider every row of the cost matrix individually and find the difference between to least cost cells in it. Repeat this step for every column. Consider all cost differences and identify the row or column with the largest value. Now select the cell with the smallest cost in the row or column so chosen and make allocation in that. The number of units allocated would be lower of the corresponding supply and demand values. In case of a tie in the largest cost difference, although either of them may be chosen, it is preferable to choose the cost difference corresponding to which the largest number of units may be assigned or corresponding to which the cell chosen has minimum cost.

Delete the row or column which has been satisfied by the allocation and adjust the quantity of demand/supply. Recalculate the cost differences for the reduced matrix and follow the above mentioned steps. Continue with this process until all units have been assigned.

Example: **A firm owns facilities at seven places. It has manufacturing plants at places A,B and C with daily output of 500,300 and 200 units of an item respectively. It has warehouse at places P,Q,R and S with daily requirements of 180,150,350 and 320 units respectively. Per unit shipping charges on different routes are given below:**

To:	*P*	*Q*	*R*	*S*
From A:	12	10	12	13
From B:	7	11	8	14
From C:	6	16	11	7

The firm wants to send the output from various plants to warehouses involving minimum transportation cost. How should it route the product so as to achieve its objective?

Solution: First of all we formulate the initial transportation tableau for the above problem and then implement the Vogel's approximation method to obtain the initial feasible solution as discussed below:

Iteration 1: The cost differences between the pairs of least cost cells are taken for each row and column. Based on the result of cost differences, column designated S is selected as it represents the largest difference 6(=13-7). Furthermore the lowest cost cell CS of the column is assigned a value 200and row C is deleted. The requirement at warehouse S is revised at 120.

Iteration 2: The cost differences are recalculated. The largest value is for column P and the least cost cell is BP. Considering supply and demand, 180 is allocated to the BP cell and column P is deleted. Also the supply at B is revised down to 120.

Iteration 3: The highest cost difference is 4 which in respect of column R. Quantity 120 is allocated to the least cost cell BR and row B is deleted. The column total is put at 230. Finally only one row is left and therefore no cost differences need to be recalculated. The allocations for each of the cells remaining are made having reference to the requirements: 150 to AQ, 230 to AR and 120 to AS. The entire process is depicted in the below table:

Note: When a row is deleted the other row differences do not change and the column cost differences need to be recalculated, and if a column is deleted, then the row differences require recalculation. In event of both row and column being deleted (when $a_i=b_j$) then all cost differences need to be calculated again.

To → / From ↓	P	Q	R	S	Supply	Iteration I	Iteration II	Iteration III
A	12	10 [150]	12 [230]	13 [120]	500	2	2	2
B	7 [180]	11	8 [120]	14	~~300~~ 120	1	1	3
C	6	16	11	7 [200]	200	1	-	-
Demand	180	150	~~350~~ 230	~~320~~ 120	1000			
I	1	1	3	(6)				
II	(5)	1	4	1				
III	-	1	(4)	1				

The initial feasible solution is reproduced in the following table:

To → / From ↓	P	Q	R	S	Supply
A	12	10 [150]	12 [230]	13 [120]	500
B	7 [180]	11	8 [120]	14	300
C	6	16	11	7 [200]	200
Demand	180	150	350	320	1000

$$Total\ cost = 10\times150+12\times230+13\times120+7\times180+8\times120+7\times200 = Rs\ 9,440$$

The following points may be noted:

- The VAM is also called penalty method because the cost differences that it uses are nothing but the penalties of not using the least cost routes. Since the objective function is the minimization of the transportation cost, in each iteration that route is selected which involves the maximum penalty of not being used.
- The initial feasible solution obtained for the above mentioned examples using three different methods shows that VAMs performance was the best of all as it involves the lowest cost total among all the three initial solutions.

Step 2: Testing the optimality.

After obtaining the initial basic feasible solution, the next step is to test whether it is optimal or not. There are two methods of testing the optimality of a basic feasible solution. They are the stepping stone method in which optimality test is applied by calculating the opportunity cost of each empty cell and other is the MODI (modified distribution method).

Step 3: Improving the solution.

By applying either of the methods , if the solution is found to be optimal, then the process terminates. If the solution is non optimal then a revised and improved basic feasible solution is obtained. This is done by transferring units from an occupied cell to an empty cell that has the largest opportunity cost, and then adjusting the units in other related cells in a way that all the supply and demand requirements are satisfied.

The solution so obtained is again tested for optimality and revised if necessary.

UNBALANCED TRANSPORTATION PROBLEMS

Unbalanced transportation problems are a class of problems where either the aggregate supply exceeds the aggregate demand or the aggregate supply falls short of the aggregate demand. Such problems can be solved only after they are balanced. This is done as follows:

When the aggregate supply exceeds the aggregate demand, the excess supply is assumed to go to inventory and costs nothing for shipping. A column of slack variables is added to the transportation tableau which represents a dummy destination with a requirement equal to the amount of excess supply and the transportation costs equal to 0. On the other hand, when the aggregate demand exceeds the aggregate supply in a transportation problem, balance is restored by adding a dummy origin. The row representing it is added with an assumed total availability equal to the difference between the total demand and supply, and with each of the cells having a zero unit cost. In some cases, however, when the penalty of not satisfying the demand at a particular destination(s) is given, then such penalty value should be considered and not zero.

Once the transportation problem is balanced, its solution proceeds in the same manner as discussed in the above sections.

Example: A company has three plants at locations A,B and C which produce the same product. It has to supply this to buyers located at D,E and F. The weekly plant capacities for A,B and C are 100, 800 and 150 units respectively, while the buyer requirements are 750, 200 and 500 units respectively for D,E and F. The unit shipping costs in (Rs) are given here:

	Buyer		
Plant	D	E	F
A	8	4	10
B	9	7	9
C	6	5	8

Assume that then penalty for failing to supply buyer requirement is Rs 4, Rs 3,and Rs 3 per unit in respect of D,E and F respectively.

Determine the optimal distribution for the company so as to minimize the cost of transportation and penalty payable.

Solution: Since this is an unbalanced transportation problem with aggregate requirement being Rs 1450 and aggregate supply being 1050 units. We will first balance the problem by adding a dummy plant T with supply value of 400 units and cost elements equal to the given penalty values 4,3 and 3 in column D,E and F respectively. This information along with the initial solution to the problem using VAM is given in the following table.

Plant	Buyer			Supply	u_i
	D	E	F		
A	8 (-2)	[100] 4	10 (-4)	100	0
B	[600] 9	[100] 7	[100] 9	800	3
C	[150] 6	5 (-1)	8 (-2)	150	0
T	4 (-1)	3 (-2)	[400] 3	400	-3
Demand	750	200	500	1450	
v_j	6	4	6		

$$Total\ cost = 4\times100 + 9\times600 + 7\times100 + 9\times100 + 6\times150 = Rs8,300$$

$$Penalty = 3\times400 = Rs1200$$

EXERCISE:

1. Determine the optimal solution to the problem given below. Obtain the initial solution by VAM and determine the optimality by MODI method.

		To Market				
		M_1	M_2	M_3	M_4	Supply
From Plant	P_1	6	4	9	1	40
	P_2	20	6	11	3	40
	P_3	7	1	0	14	50
	P_4	7	1	12	6	90
Demand		90	30	50	30	

2. Solve the following problem using transportation method, obtaining the initial feasible solution by VAM. The cell entries in the table are unit costs (in rupees) Also determine whether the solution is optimal or not.

From	**To**					
	1	2	3	4	5	Supply
1	80	69	103	64	61	12
2	47	100	72	65	40	16
3	16	103	87	36	94	20
4	86	15	57	19	25	8
5	27	20	72	94	19	8
Demand	16	14	18	6	10	