3

Tours and Matchings

3.1 Eulerian graphs

The first proper problem in graph theory was the Königsberg bridge problem. In general, this problem concerns of travelling in a graph such that one tries to avoid using any edge twice. In practice these eulerian problems occur, for instance, in optimizing distribution networks – such as delivering mail, where in order to save time each street should be travelled only once. The same problem occurs in mechanical graph plotting, where one avoids lifting the pen off the paper while drawing the lines.

Euler tours

DEFINITION. A walk $W = e_1 e_2 \dots e_n$ is a **trail**, if $e_i \neq e_j$ for all $i \neq j$. An **Euler trail** of a graph G is a trail that visits every edge once. A connected graph G is **eulerian**, if it has a closed trail containing every edge of G. Such a trail is called an **Euler tour**.

Notice that if $W = e_1 e_2 \dots e_n$ is an Euler tour (and so $E_G = \{e_1, e_2, \dots, e_n\}$), also $e_i e_{i+1} \dots e_n e_1 \dots e_{i-1}$ is an Euler tour for all $i \in [1, n]$. A complete proof of the following **Euler's Theorem** was first given by HIERHOLZER in 1873.

Theorem 3.1 (EULER **(1736),** HIERHOLZER **(1873)).** *A connected graph G is eulerian if and only if every vertex has an even degree.*

Proof. ($\Rightarrow$) Suppose $W\colon u \xrightarrow{\star} u$ is an Euler tour. Let v ($\neq u$) be a vertex that occurs k times in W. Every time an edge arrives at v, another edge departs from v, and therefore $d_G(v) = 2k$. Also, $d_G(u)$ is even, since W starts and ends at u.

($\Leftarrow$) Assume G is a nontrivial connected graph such that $d_G(v)$ is even for all $v \in G$. Let

$$W = e_1 e_2 \dots e_n \colon v_0 \xrightarrow{\star} v_n \quad \text{with} \quad e_i = v_{i-1} v_i$$

be a longest trail in G. It follows that all $e = v_n w \in G$ are among the edges of W, for, otherwise, W could be prolonged to We. In particular, $v_0 = v_n$, that is, W is a closed trail. (Indeed, if it were $v_n \neq v_0$ and v_n occurs k times in W, then $d_G(v_n) = 2(k-1)+1$ and that would be odd.)

If W is not an Euler tour, then, since G is connected, there exists an edge $f = v_i u \in G$ for some i, which is not in W. However, now

$$e_{i+1} \dots e_n e_1 \dots e_i f$$

is a trail in G, and it is longer than W. This contradiction to the choice of W proves the claim. □

Example 3.1. The k-cube Q_k is eulerian for even integers k, because Q_k is k-regular.

Theorem 3.2. *A connected graph has an Euler trail if and only if it has at most two vertices of odd degree.*

Proof. If G has an Euler trail $u \xrightarrow{\star} v$, then, as in the proof of Theorem 3.1, each vertex $w \notin \{u, v\}$ has an even degree.

Assume then that G is connected and has at most two vertices of odd degree. If G has no vertices of odd degree then, by Theorem 3.1, G has an Euler trail. Otherwise, by the handshaking lemma, every graph has an even number of vertices with odd degree, and therefore G has exactly two such vertices, say u and v. Let H be a graph obtained from G by adding a vertex w, and the edges uw and vw. In H every vertex has an even degree, and hence it has an Euler tour, say $u \xrightarrow{\star} v \to w \to u$. Here the beginning part $u \xrightarrow{\star} v$ is an Euler trail of G. □

The Chinese postman

The following problem is due to GUAN MEIGU (1962). Consider a village, where a postman wishes to plan his route to save the legs, but still every street has to be walked through. This problem is akin to Euler's problem and to the shortest path problem.

Let G be a graph with a weight function $\alpha \colon E_G \to \mathbb{R}^+$. The **Chinese postman problem** is to find a minimum weighted tour in G (starting from a given vertex, the post office).

If G is *eulerian*, then any Euler tour will do as a solution, because such a tour traverses each edge exactly once and this is the best one can do. In this case the weight of the optimal tour is the total weight of the graph G, and there is a good algorithm for finding such a tour:

Fleury's algorithm:

- Let $v_0 \in G$ be a chosen vertex, and let W_0 be the trivial path on v_0.
- Repeat the following procedure for $i = 1, 2, \dots$ as long as possible: suppose a trail $W_i = e_1 e_2 \dots e_i$ has been constructed, where $e_j = v_{j-1}v_j$.
  Choose an edge e_{i+1} ($\neq e_j$ for $j \in [1, i]$) so that

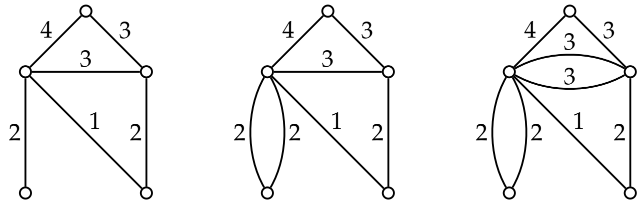
(i) e_{i+1} has an end v_i, and
(ii) e_{i+1} is not a bridge of $G_i = G - \{e_1, \dots, e_i\}$, unless there is no alternative.

Notice that, as is natural, the weights $\alpha(e)$ play no role in the eulerian case.

Theorem 3.3. *If G is eulerian, then any trail of G constructed by Fleury's algorithm is an Euler tour of G.*

Proof. Exercise. □

If G is *not eulerian*, the poor postman has to walk at least one street twice. This happens, *e.g.*, if one of the streets is a dead end, and in general if there is a street corner of an odd number of streets. We can attack this case by reducing it to the eulerian case as follows. An edge $e = uv$ will be **duplicated**, if it is added to G parallel to an existing edge $e' = uv$ with the same weight, $\alpha(e') = \alpha(e)$.



Above we have duplicated two edges. The rightmost multigraph is eulerian.

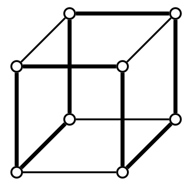
There is a good algorithm by EDMONDS AND JOHNSON (1973) for the construction of an optimal eulerian supergraph by duplications. Unfortunately, this algorithm is somewhat complicated, and we shall skip it.

3.2 Hamiltonian graphs

In the connector problem we reduced the cost of a spanning graph to its minimum. There are different problems, where the cost is measured by an active user of the graph. For instance, in the **travelling salesman problem** a person is supposed to visit each town in his district, and this he should do in such a way that saves time and money. Obviously, he should plan the travel so as to visit each town once, and so that the overall flight time is as short as possible. In terms of graphs, he is looking for a minimum weighted Hamilton cycle of a graph, the vertices of which are the towns and the weights on the edges are the flight times. Unlike for the shortest path and the connector problems no efficient reliable algorithm is known for the travelling salesman problem. Indeed, it is widely believed that no practical algorithm exists for this problem.

Hamilton cycles

DEFINITION. A path P of a graph G is a **Hamilton path**, if P visits every vertex of G once. Similarly, a cycle C is a **Hamilton cycle**, if it visits each vertex once. A graph is **hamiltonian**, if it has a Hamilton cycle.



Note that if $C : u_1 \to u_2 \to \cdots \to u_n$ is a Hamilton cycle, then so is $u_i \to \dots u_n \to u_1 \to \dots u_{i-1}$ for each $i \in [1, n]$, and thus we can choose where to start the cycle.

Example 3.2. It is obvious that each K_n is hamiltonian whenever $n \geq 3$. Also, as is easily seen, $K_{n,m}$ is hamiltonian if and only if $n = m \geq 2$. Indeed, let $K_{n,m}$ have a bipartition (X, Y), where $|X| = n$ and $|Y| = m$. Now, each cycle in $K_{n,m}$ has even length as the graph is bipartite, and thus the cycle visits the sets X, Y equally many times, since X and Y are stable subsets. But then necessarily $|X| = |Y|$.

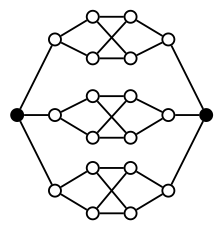
Unlike for eulerian graphs (Theorem 3.1) no good characterization is known for hamiltonian graphs. Indeed, the problem to determine if G is hamiltonian is NP-complete. There are, however, some interesting general conditions.

Lemma 3.1. *If G is hamiltonian, then for every nonempty subset $S \subseteq V_G$,*

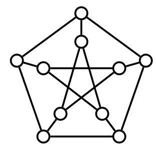
$$c(G{-}S) \leq |S|\,.$$

Proof. Let $\emptyset \neq S \subseteq V_G$, $u \in S$, and let $C\colon u \xrightarrow{\star} u$ be a Hamilton cycle of G. Assume $G{-}S$ has k connected components, G_i, $i \in [1, k]$. The case $k = 1$ is trivial, and hence suppose that $k > 1$. Let u_i be the last vertex of C that belongs to G_i, and let v_i be the vertex that follows u_i in C. Now $v_i \in S$ for each i by the choice of u_i, and $v_j \neq v_t$ for all $j \neq t$, because C is a cycle and $u_i v_i \in G$ for all i. Thus $|S| \geq k$ as required. □

Example 3.3. Consider the graph on the right. In G, $c(G{-}S) = 3 > 2 = |S|$ for the set S of black vertices. Therefore G does not satisfy the condition of Lemma 3.1, and hence it is not hamiltonian. Interestingly this graph is (X, Y)-bipartite of even order with $|X| = |Y|$. It is also 3-regular.



Example 3.4. Consider the **Petersen graph** on the right, which appears in many places in graph theory as a counter example for various conditions. This graph is not hamiltonian, but it does satisfy the condition $c(G{-}S) \leq |S|$ for all $S \neq \emptyset$. Therefore the conclusion of Lemma 3.1 is *not sufficient* to ensure that a graph is hamiltonian.



The following theorem, due to ORE, generalizes an earlier result by DIRAC (1952).

Theorem 3.4 (ORE (1962)). *Let G be a graph of order $\nu_G \geq 3$, and let $u, v \in G$ be such that*

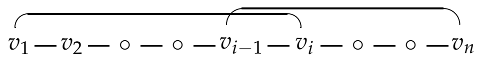
$$d_G(u) + d_G(v) \geq \nu_G\,.$$

Then G is hamiltonian if and only if $G + uv$ is hamiltonian.

Proof. Denote $n = \nu_G$. Let $u, v \in G$ be such that $d_G(u) + d_G(v) \geq n$. If $uv \in G$, then there is nothing to prove. Assume thus that $uv \notin G$.

($\Rightarrow$) This is trivial since if G has a Hamilton cycle C, then C is also a Hamilton cycle of $G + uv$.

($\Leftarrow$) Denote $e = uv$ and suppose that $G + e$ has a Hamilton cycle C. If C does not use the edge e, then it is a Hamilton cycle of G. Suppose thus that e is on C. We may then assume that $C\colon u \xrightarrow{\star} v \to u$. Now $u = v_1 \to v_2 \to \ldots \to v_n = v$ is a Hamilton *path* of G. There exists an i with $1 < i < n$ such that $uv_i \in G$ and $v_{i-1}v \in G$. For, otherwise, $d_G(v) < n - d_G(u)$ would contradict the assumption.



But now $u = v_1 \xrightarrow{\star} v_{i-1} \to v_n \to v_{n-1} \xrightarrow{\star} v_{i+1} \to v_i \to v_1 = u$ is a Hamilton cycle in G. □

Closure

DEFINITION. For a graph G, define inductively a sequence $G_0, G_1, \dots, G_k$ of graphs such that

$$G_0 = G \text{ and } G_{i+1} = G_i + uv\,,$$

where u and v are any vertices such that $uv \notin G_i$ and $d_{G_i}(u) + d_{G_i}(v) \geq \nu_G$. This procedure stops when no new edges can be added to G_k for some k, that is, in G_k, for all $u, v \in G$ either $uv \in G_k$ or $d_{G_k}(u) + d_{G_k}(v) < \nu_G$. The result of this procedure is the **closure** of G, and it is denoted by $cl(G)$ $(= G_k)$.

In each step of the construction of $cl(G)$ there are usually alternatives which edge uv is to be added to the graph, and therefore the above procedure is not deterministic. However, the *final result* $cl(G)$ is independent of the choices.

Lemma 3.2. *The closure $cl(G)$ is uniquely defined for all graphs G of order $\nu_G \geq 3$.*

Proof. Denote $n = \nu_G$. Suppose there are two ways to close G, say

$$H = G + \{e_1, \dots, e_r\} \text{ and } H' = G + \{f_1, \dots, f_s\}\,,$$

where the edges are added in the given orders. Let $H_i = G + \{e_1, \dots, e_i\}$ and $H'_i = G + \{f_1, \dots, f_i\}$. For the initial values, we have $G = H_0 = H'_0$. Let $e_k = uv$ be the first edge such that $e_k \neq f_i$ for all i. Then $d_{H_{k-1}}(u) + d_{H_{k-1}}(v) \geq n$, since $e_k \in H_k$, but $e_k \notin H_{k-1}$. By the choice of e_k, we have $H_{k-1} \subseteq H'$, and thus also $d_{H'}(u) + d_{H'}(v) \geq n$, which means that $e = uv$ must be in H'; a contradiction. Therefore $H \subseteq H'$. Symmetrically, we deduce that $H' \subseteq H$, and hence $H' = H$. □

Theorem 3.5. *Let G be a graph of order $\nu_G \geq 3$.*

(i) G is hamiltonian if and only if its closure $cl(G)$ is hamiltonian.
(ii) If $cl(G)$ is a complete graph, then G is hamiltonian.

Proof. First, $G \subseteq cl(G)$ and G spans $cl(G)$, and thus if G is hamiltonian, so is $cl(G)$.

In the other direction, let $G = G_0, G_1, \dots, G_k = cl(G)$ be a construction sequence of the closure of G. If $cl(G)$ is hamiltonian, then so are $G_{k-1}, \dots, G_1$ and G_0 by Theorem 3.4.

The Claim (ii) follows from (i), since each complete graph is hamiltonian. □

Theorem 3.6. *Let G be a graph of order $\nu_G \geq 3$. Suppose that for all nonadjacent vertices u and v, $d_G(u) + d_G(v) \geq \nu_G$. Then G is hamiltonian. In particular, if $\delta(G) \geq \frac{1}{2}\nu_G$, then G is hamiltonian.*

Proof. Since $d_G(u) + d_G(v) \geq \nu_G$ for all nonadjacent vertices, we have $cl(G) = K_n$ for $n = \nu_G$, and thus G is hamiltonian. The second claim is immediate, since now $d_G(u) + d_G(v) \geq \nu_G$ for all $u, v \in G$ whether adjacent or not. □

Chvátal's condition

The hamiltonian problem of graphs has attracted much attention, at least partly because the problem has practical significance. (Indeed, the first example where DNA computing was applied, was the hamiltonian problem.)

There are some general improvements of the previous results of this chapter, and quite many improvements in various special cases, where the graphs are somehow restricted. We become satisfied by two general results.

Theorem 3.7 (CHVÁTAL (1972)). *Let G be a graph with $V_G = \{v_1, v_2, \dots, v_n\}$, for $n \geq 3$, ordered so that $d_1 \leq d_2 \leq \dots \leq d_n$, for $d_i = d_G(v_i)$. If for every $i < n/2$,*

$$d_i \leq i \implies d_{n-i} \geq n - i \,, \tag{3.1}$$

then G is hamiltonian.

Proof. First of all, we may suppose that G is closed, $G = cl(G)$, because G is hamiltonian if and only if $cl(G)$ is hamiltonian, and adding edges to G does not decrease any of its degrees, that is, if G satisfies (3.1), so does $G + e$ for every e. We show that, in this case, $G = K_n$, and thus G is hamiltonian.

Assume on the contrary that $G \neq K_n$, and let $uv \notin G$ with $d_G(u) \leq d_G(v)$ be such that $d_G(u) + d_G(v)$ is as large as possible. Because G is closed, we must have $d_G(u) + d_G(v) < n$, and therefore $d_G(u) = i < n/2$. Let $A = \{w \mid vw \notin G, w \neq v\}$. By our choice, $d_G(w) \leq i$ for all $w \in A$, and, moreover,

$$|A| = (n-1) - d_G(v) \geq d_G(u) = i \,.$$

Consequently, there are at least i vertices w with $d_G(w) \leq i$, and so $d_i \leq d_G(u) = i$.

Similarly, for each vertex from $B = \{w \mid uw \notin G, w \neq u\}$, $d_G(w) \leq d_G(v) < n - d_G(u) = n - i$, and

$$|B| = (n-1) - d_G(u) = (n-1) - i \,.$$

Also $d_G(u) < n - i$, and thus there are at least $n - i$ vertices w with $d_G(w) < n - i$. Consequently, $d_{n-i} < n - i$. This contradicts the obtained bound $d_i \leq i$ and the condition (3.1). □

Note that the condition (3.1) is easily checkable for any given graph.

3.3 Matchings

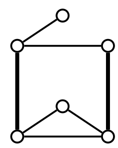
In matching problems we are given an availability relation between the elements of a set. The problem is then to find a pairing of the elements so that each element is paired (matched) uniquely with an available companion.

A special case of the matching problem is the **marriage problem**, which is stated as follows. Given a set X of boys and a set Y of girls, under what condition can each boy marry a girl who cares to marry him? This problem has many variations. One of them is the **job assignment problem**, where we are given n applicants and m jobs, and we should assign each applicant to a job he is qualified. The problem is that an applicant may be qualified for several jobs, and a job may be suited for several applicants.

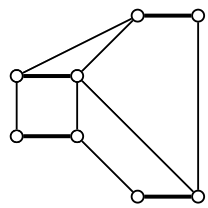
Maximum matchings

DEFINITION. For a graph G, a subset $M \subseteq E_G$ is a **matching** of G, if M contains no adjacent edges. The two ends of an edge $e \in M$ are **matched under** M. A matching M is a **maximum matching**, if for no matching M', $|M| < |M'|$.

The two vertical edges on the right constitute a matching M that is *not a maximum matching*, although you cannot add any edges to M to form a larger matching. This matching is not maximum because the graph has a matching of three edges.



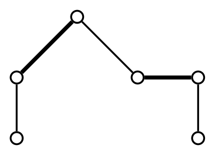
DEFINITION. A matching M **saturates** $v \in G$, if v is an end of an edge in M. Also, M **saturates** $A \subseteq V_G$, if it saturates every $v \in A$. If M saturates V_G, then M is a **perfect matching**.



It is clear that every perfect matching is maximum. On the right the horizontal edges form a perfect matching.

DEFINITION. Let M be a matching of G. An odd path $P = e_1e_2 \dots e_{2k+1}$ is M-**augmented**, if

- P alternates between $E_G \setminus M$ and M (that is, $e_{2i+1} \in G - M$ and $e_{2i} \in M$), and
- the ends of P are not saturated.



Lemma 3.3. *If G is connected with $\Delta(G) \leq 2$, then G is a path or a cycle.*

Proof. Exercise. □

We start with a result that gives a necessary and sufficient condition for a matching to be maximum. One can use the first part of the proof to construct a maximum matching in an iterative manner starting from any matching M and from any M-augmented path.

Theorem 3.8 (BERGE **(1957)).** *A matching M of G is a maximum matching if and only if there are no M-augmented paths in G.*

Proof. ($\Rightarrow$) Let a matching M have an M-augmented path $P = e_1 e_2 \dots e_{2k+1}$ in G. Here $e_2, e_4, \dots, e_{2k} \in M$, $e_1, e_3, \dots, e_{2k+1} \notin M$. Define $N \subseteq E_G$ by

$$N = (M \setminus \{e_{2i} \mid i \in [1,k]\}) \cup \{e_{2i+1} \mid i \in [0,k]\}.$$

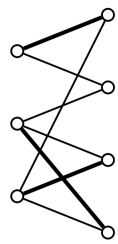
Now, N is a matching of G, and $|N| = |M| + 1$. Therefore M is not a maximum matching.

($\Leftarrow$) Assume N is a maximum matching, but M is not. Hence $|N| > |M|$. Consider the subgraph $H = G[M \triangle N]$ for the symmetric difference $M \triangle N$. We have $d_H(v) \leq 2$ for each $v \in H$, because v is an end of at most one edge in M and N. By Lemma 3.3, each connected component A of H is either a path or a cycle.

Since no $v \in A$ can be an end of two edges from N or from M, each connected component (path or a cycle) A alternates between N and M. Now, since $|N| > |M|$, there is a connected component A of H, which has more edges from N than from M. This A cannot be a cycle, because an alternating cycle has even length, and it thus contains equally many edges from N and M. Hence $A\colon u \xrightarrow{\star} v$ is a path (of odd length), which starts and ends with an edge from N. Because A is a connected component of H, the ends u and v are not saturated by M, and, consequently, A is an M-augmented path. This proves the theorem. □

Example 3.5. Consider the k-cube Q_k for $k \geq 1$. Each maximum matching of Q_k has 2^{k-1} edges. Indeed, the matching $M = \{(0u, 1u) \mid u \in \mathbb{B}^{k-1}\}$, has 2^{k-1} edges, and it is clearly perfect.

Hall's theorem



For a subset $S \subseteq V_G$ of a graph G, denote

$$N_G(S) = \{v \mid uv \in G \text{ for some } u \in S\}.$$

If G is (X,Y)-bipartite, and $S \subseteq X$, then $N_G(S) \subseteq Y$.

The following result, known as the

Theorem 3.9 (HALL **(1935)).** *Let G be a (X,Y)-bipartite graph. Then G contains a matching M saturating X if and only if*

$$|S| \leq |N_G(S)| \quad \textit{for all } S \subseteq X. \tag{3.2}$$

Proof. ($\Rightarrow$) Let M be a matching that saturates X. If $|S| > |N_G(S)|$ for some $S \subseteq X$, then not all $x \in S$ can be matched with different $y \in N_G(S)$.

($\Leftarrow$) Let G satisfy Hall's condition (3.2). We prove the claim by induction on $|X|$.

If $|X| = 1$, then the claim is clear. Let then $|X| \geq 2$, and assume (3.2) implies the existence of a matching that saturates every proper subset of X.

If $|N_G(S)| \geq |S| + 1$ for every nonempty $S \subseteq X$ with $S \neq X$, then choose an edge $uv \in G$ with $u \in X$, and consider the induced subgraph $H = G - \{u, v\}$. For all $S \subseteq X \setminus \{u\}$, $|N_H(S)| \geq |N_G(S)| - 1 \geq |S|$, and hence, by the induction hypothesis, H contains a matching M saturating $X \setminus \{u\}$. Now $M \cup \{uv\}$ is a matching saturating X in G, as was required.

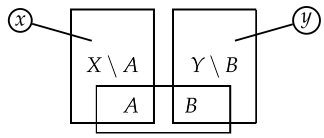
Suppose then that there exists a nonempty subset $R \subseteq X$ with $R \neq X$ such that $|N_G(R)| = |R|$. The induced subgraph $H_1 = G[R \cup N_G(R)]$ satisfies (3.2) (since G does), and hence, by the induction hypothesis, H_1 contains a matching M_1 that saturates R (with the other ends in $N_G(R)$).

Also, the induced subgraph $H_2 = G[V_G \setminus A]$, for $A = R \cup N_G(R)$, satisfies (3.2). Indeed, if there were a subset $S \subseteq X \setminus R$ such that $|N_{H_2}(S)| < |S|$, then we would have

$$|N_G(S \cup R)| = |N_{H_2}(S)| + |N_{H_1}(R)| < |S| + |N_G(R)| = |S| + |R| = |S \cup R|$$

(since $S \cap R = \emptyset$), which contradicts (3.2) for G. By the induction hypothesis, H_2 has a matching M_2 that saturates $X \setminus R$ (with the other ends in $Y \setminus N_G(R)$). Combining the matchings for H_1 and H_2, we get a matching $M_1 \cup M_2$ saturating X in G. □

Second proof. This proof of the direction ($\Leftarrow$) uses Menger's theorem. Let H be the graph obtained from G by adding two new vertices x, y such that x is adjacent to each $v \in X$ and y is adjacent to each $v \in Y$. There exists a matching saturating X if (and only if) the number of independent paths $x \xrightarrow{\star} y$ is equal to $|X|$. For this, by Menger's theorem, it suffices to show that every set S that separates x and y in H has at least $|X|$ vertices.



Let $S = A \cup B$, where $A \subseteq X$ and $B \subseteq Y$. Now, vertices in $X \setminus A$ are not adjacent to vertices of $Y \setminus B$, and hence we have $N_G(X \setminus A) \subseteq B$, and thus that $|X \setminus A| \leq |N_G(X \setminus A)| \leq |B|$ using the condition (3.2).

We conclude that $|S| = |A| + |B| \geq |X|$. □

Corollary 3.1 (FROBENIUS (1917)). *If G is a k-regular bipartite graph with $k > 0$, then G has a perfect matching.*

Proof. Let G be k-regular (X, Y)-bipartite graph. By regularity, $k \cdot |X| = \varepsilon_G = k \cdot |Y|$, and hence $|X| = |Y|$. Let $S \subseteq X$. Denote by E_1 the set of the edges with an end in S, and by E_2 the set of the edges with an end in $N_G(S)$. Clearly, $E_1 \subseteq E_2$. Therefore, $k \cdot |N_G(S)| = |E_2| \geq |E_1| = k \cdot |S|$, and so $|N_G(S)| \geq |S|$. By Theorem 3.9, G has a matching that saturates X. Since $|X| = |Y|$, this matching is necessarily perfect. □