2

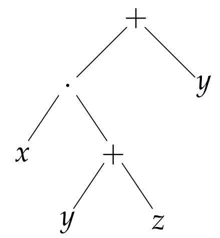
Connectivity of Graphs

2.1 Bipartite graphs and trees

In problems such as the shortest path problem we look for minimum solutions that satisfy the given requirements. The solutions in these cases are usually subgraphs without cycles. Such connected graphs will be called trees, and they are used, *e.g.*, in search algorithms for databases. For concrete applications in this respect, see

T.H. CORMEN, C.E. LEISERSON AND R.L. RIVEST, "Introduction to Algorithms", MIT Press, 1993.

Certain structures with operations are representable as trees. These trees are sometimes called *construction trees*, *decomposition trees*, *factorization trees* or *grammatical trees*. Grammatical trees occur especially in linguistics, where syntactic structures of sentences are analyzed. On the right there is a tree of operations for the arithmetic formula $x \cdot (y + z) + y$.



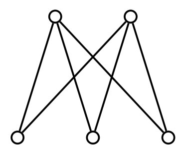
Bipartite graphs

DEFINITION. A graph G is called **bipartite**, if V_G has a partition to two subsets X and Y such that each edge $uv \in G$ connects a vertex of X and a vertex of Y. In this case, (X, Y) is a **bipartition** of G, and G is (X, Y)-**bipartite**.

A bipartite graph G (as in the above) is **complete** (m, k)-**bipartite**, if $|X| = m$, $|Y| = k$, and $uv \in G$ for all $u \in X$ and $v \in Y$.

All complete (m, k)-bipartite graphs are isomorphic. Let $K_{m,k}$ denote such a graph.

A subset $X \subseteq V_G$ is **stable**, if $G[X]$ is a discrete graph.



$K_{2,3}$

The following result is clear from the definitions.

Theorem 2.1. *A graph G is bipartite if and only if V_G has a partition to two stable subsets.*

Example 2.1. The k-cube Q_k of Example 1.5 is bipartite for all k. Indeed, consider $A = \{u \mid u \text{ has an even number of } 1's\}$ and $B = \{u \mid u \text{ has an odd number of } 1's\}$. Clearly, these sets partition $\mathbb{B}^k$, and they are stable in Q_k.

Theorem 2.2. *A graph G is bipartite if and only if G it has no odd cycles (as subgraph).*

Proof. ($\Rightarrow$) Observe that if G is (X,Y)-bipartite, then so are all its subgraphs. However, an odd cycle C_{2k+1} is not bipartite.

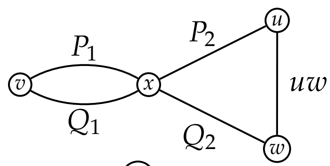
($\Leftarrow$) Suppose that all cycles in G are even. First, we note that it suffices to show the claim for connected graphs. Indeed, if G is disconnected, then each cycle of G is contained in one of the connected components $G_1, \dots, G_p$ of G. If G_i is (X_i, Y_i)-bipartite, then G has the bipartition $(X_1 \cup X_2 \cup \cdots \cup X_p, Y_1 \cup Y_2 \cup \cdots \cup Y_p)$.

Assume thus that G is connected. Let $v \in G$ be a chosen vertex, and define

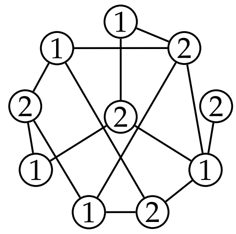
$$X = \{x \mid d_G(v,x) \text{ is even}\} \quad \text{and} \quad Y = \{y \mid d_G(v,y) \text{ is odd}\}.$$

Since G is connected, $V_G = X \cup Y$. Also, by the definition of distance, $X \cap Y = \emptyset$.

Let then $u, w \in G$ be both in X or both in Y, and let $P\colon v \xrightarrow{\star} u$ and $Q\colon v \xrightarrow{\star} w$ be (among the) shortest paths from v to u and w. Assume that x is the last common vertex of P and Q: $P = P_1P_2$, $Q = Q_1Q_2$, where $P_2\colon x \xrightarrow{\star} u$ and $Q_2\colon x \xrightarrow{\star} w$ are independent. Since P and Q are shortest paths, P_1 and Q_1 are shortest paths $v \xrightarrow{\star} x$. Consequently, $|P_1| = |Q_1|$.



Thus $|P_2|$ and $|Q_2|$ have the same parity and hence the sum $|P_2| + |Q_2|$ is even, i.e., the path $P_2^{-1}Q_2$ is even, and so $uw \notin E_G$ by assumption. Therefore X and Y are stable subsets, and G is bipartite as claimed. □



Checking whether a graph is bipartite is easy. Indeed, this can be done by using two 'opposite' colours, say 1 and 2. Start from any vertex v_1, and colour it by 1. Then colour the neighbours of v_1 by 2, and proceed by colouring all neighbours of an already coloured vertex by the opposite colour.

If the whole graph can be coloured without contradiction, then G is (X,Y)-bipartite, where X consists of those vertices with colour 1, and Y of those vertices with colour 2; otherwise, at some point one of the vertices gets both colours, and in this case, G is not bipartite.

Example 2.2 (ERDÖS **(1965)).** We show that each graph G has a bipartite subgraph $H \subseteq G$ such that $\varepsilon_H \geq \frac{1}{2}\varepsilon_G$. Indeed, let $V_G = X \cup Y$ be a partition such that the number of edges between X and Y is maximum. Denote

$$F = E_G \cap \{uv \mid u \in X, v \in Y\},$$

and let $H = G[F]$. Obviously H is a spanning subgraph, and it is bipartite.

By the maximum condition, $d_H(v) \geq d_G(v)/2$, since, otherwise, v is on the wrong side. (That is, if $v \in X$, then the pair $X' = X \setminus \{v\}$, $Y' = Y \cup \{v\}$ does better that the pair X, Y.) Now

$$\varepsilon_H = \frac{1}{2}\sum_{v \in H} d_H(v) \geq \frac{1}{2}\sum_{v \in G}\frac{1}{2}d_G(v) = \frac{1}{2}\varepsilon_G\,.$$