

## Operations Research: Introduction

### Origin and Development

- Term or coined in 1940 by Meclosky & Trefthan in U.K.
- came into existence during World War II for military projects for solving strategic and tactical problems. It was successful and showed remarkable progress.
- After the war it attracted attention of industrial manages for solutions to complex executive type problems.
- Then it gained importance in the field of economics, commerce and Engineering.

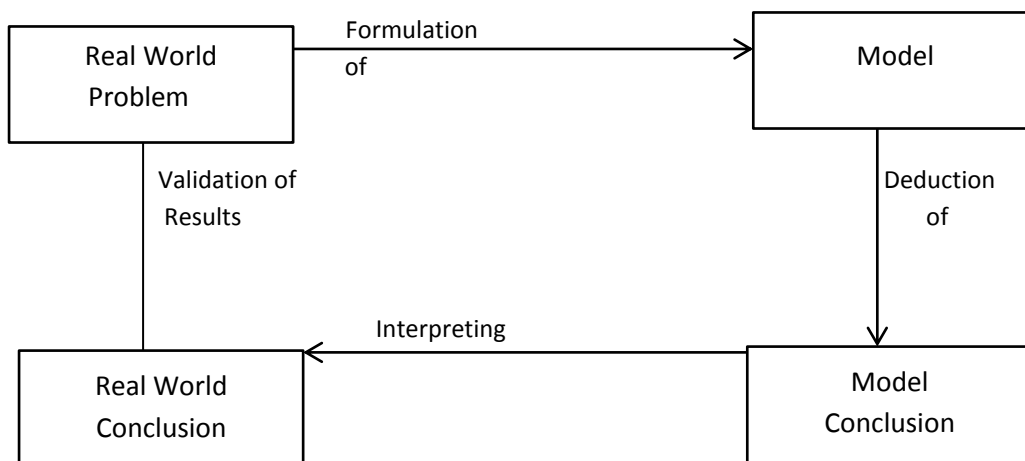
### Features

- Application of Scientific methods, tools and techniques for finding optimum solution to problems involving the operations of a system.
- Determines the best utilization of limited resources.
- Applied in decision making problems

### Concept of a Model

- Model is abstraction of reality
- Examples
  - Road map of a city to trace the shortest route from a given source to a given destination.
  - 3D view of a factory to plan the material movements in its shop floor.
  - Linear equation to forecast the demand of a product.

### Steps of Modelling



**Figure: Schematic representation of modeling steps**

Step1: Identify the management decision problem of real world

Step2. Formulate a model for the real world problem.

- a) Identify the parameters and variable which are involved in management decision problem
- b) Select the variable that appear to be most influential so that the model may be kept as simple as possible.
- c) State relationships among the variables based upon known principles

- d) Construct a model by confining all real world relationships into a system of symbolic relationships.
- Step3. Solve a set of equations, iterating set of steps or make statistical analysis to optimize performance and draw model conclusions.
- Step 4. Interpret the model conclusions in terms of real world problem's characteristics.
- Step 5. Test & validate the results.
- Step6. Implement the result.
- Step 7. Revise the model as & when necessary.

## **Scope of Operations Research**

### 1. Defence applications

- Scientific decision making for many defence operations

Examples

- Network techniques for shortest path problems
- Algorithms for vehicle routing
- Allocation techniques for shipping food grains and ammunitions.

### 2. Industrial applications

- Major four function areas
  - i. Production
    - Linear programming for aggregate planning
    - Integer programming for shop floor production scheduling
    - Network based techniques for line balancing and project management
    - Replacement analysis for equipment replacement decisions.
  - ii. Marketing
    - Linear programming for product mix problems
    - Game theory for order bidding decisions
    - Distance network related techniques for shipping finished goods.
  - iii. Personnel
    - Linear programming for manpower planning
    - Scheduling techniques for manpower scheduling.
  - iv. Finance
    - Integer programming for capital budgeting etc.

### 3. Public systems

Examples: Postal systems, banks, highways, railways, airways, hospitals, educational institutions etc.

## **Linear Programming**

- Mathematical programming technique to optimize the performance (profit/loss) under a set of constraints like machine hours, man hours, money, material etc. as specified by an organization.

### **APPLICATIONS OF LPP**

1. Product mix problem.

2. Diet planning problem.
3. Cargo loading problem.
4. Capital budgeting problem.
5. Man power planning problem.

### CONCEPT OF LPP

The model of any LPP will contain:

- Objective functions.
- Set of constraints.
- Non-negative restrictions.

Each of these components may consist of:

- Decision variable.
- Objective function coefficients.
- Technological coefficients.
- Availability of resources.

These can be explained with the following example:-

#### Product mix problem.

Determining the level of production activities carried out during a pre-decided time frame so as to gain the maximum profit.

EXAMPLE: A company manufactures two different types of products P1 and P2. Each product requires processing on milling machine and drilling machine. But each type of machine has limited hours available per week. The net profit per unit of the products, resource requirements of the products and availability of resources are given in the table:

MACHINE TYPE	PROCESSING TIME		MACHINE HOURS AVAILABLE PER WEEK
	Product P1	Product P2	
MILLING MACHINE	2	5	200
DRILLING MACHINE	4	2	240
PROFIT PER UNIT (Rs)	250	400	

Develop a LP model to determine the optimal production volume of each of the products such that the profit is maximized subject to the availability of machine hours.

**Solution:** let  $n$  be number of products to be manufactured and  $m$  be the number of machine types used.

Decision variable: is used to represent the level of achievement of a particular activity. The solution of LP will provide the optimal value for each and every decision variable of the model.

Let  $X_1$  and  $X_2$  be the production volumes of P1 and P2 respectively.

Objective function coefficient: is the constant representing the profit/unit or cost/unit of carrying out an activity.

Let  $C_1, C_2, \dots, C_n$  be the profit/unit of products P1, P2, ..., Pn respectively.

Therefore  $C_1 = 250, C_2 = 400$

Objective function: is an expression representing the total profit or cost of carrying out a set of activities at some levels:

- Benefit related objective function type → Maximization Type.
- Cost related objective function type → Minimization Type.

So, a generalized format:-

$$\text{Maximize or Minimize } Z = C_1 X_1 + C_2 X_2 + \dots + C_n X_n.$$

Here, Maximize  $Z = C_1 X_1 + C_2 X_2$

$$= 250 X_1 + 400 X_2$$

Technological coefficients ( $a_{ij}$ ): amount of resource  $i$  required for activity  $j$ , where  $i = 1$  to  $m$  and  $j = 1$  to  $n$ . so,

$a_{ij} =$

$a_{11}$	$a_{12}$	$a_{13}$	.....	$a_{1n}$
$a_{21}$				$a_{2n}$
$a_{31}$				.
.				.
.				.
.				.
.				.
.				.
.				.
$a_{m1}$	$a_{m2}$	.	.	$a_{mn}$

For the given example:

$a_{ij} =$

2	5
4	2

Resource availability: constant  $b_i$  is the amount of resource available during the planning period.

So,  $b_i =$

$b_1$
$b_2$
$b_3$
.
.
.
$b_m$

Here  $b_i =$

200
450

Set of constraints: Kind of restriction on the total amount of a particular resource requirement to carry out the activities at various levels.

$$a_{11} X_1 + a_{12} X_2 + \dots + a_{1n} X_n \leq \text{or} \geq b_1$$

$$a_{21} X_1 + a_{22} X_2 + \dots + a_{2n} X_n \leq \text{or} \geq b_2$$

.....

$$a_{m1} X_1 + a_{m2} X_2 + \dots + a_{mn} X_n \leq \text{or} \geq b_m$$

so, for the example,

$$2 X_1 + 5 X_2 \leq 200$$

$$4 X_1 + 2 X_2 \leq 240$$

Non-negative restrictions: each and every decision variable in LP model is a non- negative variable.

$$X_1, X_2, \dots, X_n \geq 0$$

So,  $X_1$  and  $X_2 \geq 0$

Therefore, complete model of the product mix problem  $\rightarrow$  Maximize  $Z = 250 X_1 + 400 X_2$

Subject to,

$$2 X_1 + 5 X_2 \leq 200$$

$$4 X_1 + 2 X_2 \leq 240, X_1 \text{ and } X_2 \geq 0$$

### ASSUMPTION IN LP

- **LINEARITY:** resource is directly proportional to the level of activity.  
Example, if time required on a particular machine for a given activity level is 5 hours/ unit, then the total amount of time required on that machine to produce 10 units of that activity is 50 hours.
- **DIVISIBILITY:** fractional values of the decision variables are permitted.
- **NON-NEGATIVITY:** decision values must be  $\geq 0$ .
- **ADDITIVITY:** total output for the given combination of activity levels is the algebraic sum of the output of each individual process.

## Properties of LP solution

1. **Feasible solution:** if all the constraints of the given LP model are satisfied by the solution of the model, feasible solution. There can be multiple feasible solutions for a given LP model.
2. **Optimal solution:** If there is no other superior solution to the solution obtained for a given LP model then the solution obtained is treated as the optimal solution.
3. **Alternate optimum solution:** For some LP model, there may be more than one combination of values of the decision variables leading to the best objective function value. Such combinations of the value of the decision variables are known as alternate optimum solution.
4. **Unbounded Solution:** for some LP model, the objective function value can be increased/ decreased infinitely without any limitation. Such solution is known as unbounded solution.
5. **infeasible solution:** -if there is no combination of values of the decision variables satisfying all the constraints of the LP model then the model is said to have infeasible solution. There is no implementable solution for the model.
6. **Degenerate solution:** redundant solution after some time.

## L P Methods

### Simplex Method

Maximize  $Z = 6x_1 + 8x_2$

Subject to

$$5x_1 + 10x_2 \leq 60$$

$$4x_1 + 4x_2 \leq 60$$

$$x_1, x_2 \geq 0$$

writing in standard form

maximize  $Z = 6x_1 + 8x_2 + 0.S_1 + 0.S_2$

Subject to

$$5x_1 + 10x_2 + S_1 = 60$$

$$4x_1 + 4x_2 + S_2 = 40$$

$$x_1, x_2 \geq 0$$

$S_1, S_2 \rightarrow$  Slack variable, introduced to balance the constraints

	$C_j$		6	8	0	0	Solution	Ratio
$CB_i$	Basic variable	$X_1$	$X_2$	$S_1$	$S_2$		$(b_j)$	
0	$S_1$	5	10	1	0		60	$60/10=6$
0	$S_2$	4	4	0	1		40	$40/4=10$
$Z_j$			0	0	0	0	0	
$C_j - Z_j$			6	8		0		

$$Z_j = \sum_{i=1}^2 (CB_i) a_{ij}$$

Max  $(C_j - Z_j) n = 8 \rightarrow$  Key Col

Min Ratio  $n = 6 \rightarrow$  Key Row

Intersection  $= 10 \rightarrow$  Key element

Optimum Condition:-

For Maximize Z, all  $C_j - Z_j$  should be  $\leq 0$

For Minimize Z, all  $C_j - Z_j$  should be  $\geq 0$

Here,  $C_j - Z_j \leq 0$

So, next iteration  $\rightarrow$

$$\text{New value} = \text{old} - \frac{\text{Col} * \text{row}}{\text{Key value}}$$

So for row 2, col  $x_1$

$$\text{New value} = 4 - 5 \times \frac{4}{10} = 2$$

Row2, Col  $S_1$

$$\text{New value} = 0 - 1 \times \frac{4}{10} = -2/5$$

Row2, Col  $S_2$

$$\text{New value} = 1 - 0 \times \frac{4}{5} = 1$$

	$C_j$	6	8	0	0	
$CB_i$	Basic Var	$X_1$	$X_2$	$S_1$	$S_2$	
8	$X_2$	$\frac{1}{2}$	1	$\frac{1}{10}$	0	6
0	$X_2$	2	0	$-\frac{2}{5}$	1	16
	$Z_j$	4	8	$\frac{4}{5}$	0	48
	$C_j - Z_j$	2	0	$-\frac{4}{5}$	-1	

All  $C_j - Z_j \leq 0$

	$C_j$	6	8	0	0		
$CB_i$	Basic Var	$X_1$	$X_2$	$S_1$	$S_2$	solution	ratio
8	$X_2$	$\frac{1}{2}$	1	$\frac{1}{10}$	0	6	
0	$S_2$	2	0	$-\frac{2}{5}$	1	16	$\frac{16}{2} = 8 \rightarrow$
	$C_j - Z_j$	2	0	$-\frac{4}{5}$	-1		

Pivotal element = 2

	$C_j$	6	8	0	0	solution
$CB_i$	Basic Var	$X_1$	$X_2$	$S_1$	$S_2$	
8	$X_2$	0	1	$\frac{1}{5}$	$-\frac{1}{4}$	2
0	$X_1$	1	0	$-\frac{1}{5}$	$\frac{1}{2}$	8
	$Z_j$	6	8	$\frac{2}{5}$	0	64
	$C_j - Z_j$	0	0	$-\frac{2}{5}$	-1	

All  $C_j - Z_j \leq 0 \rightarrow$  optimal solution

So here  $x_1 = 8$  ( production volume of P1)

$X_2 = 2$  (Prod. Volume of P2)

$Z = 64$  (Total profit in Rs)

Check by substituting in equations.

## Two phase method

- used for  $\geq =$  equality

Problem: minimize  $Z = 12 X_1 + 18 X_2 + 15 X_3$

Subject to

$$4 X_1 + 8 X_2 + 6 X_3 \geq 64$$

$$3 X_1 + 6 X_2 + 12 X_3 \geq 96$$

$$X_1, X_2, X_3 \geq 0$$

Solution writing in canonical form  $\rightarrow$

$$\text{Minimize } Z = 12 X_1 + 18 X_2 + 15 X_3 + Ma_1 + Ma_2$$

Subject to

$$4 X_1 + 8 X_2 + 6 X_3 - S_1 + a_1 = 64$$

$$3 X_1 + 6 X_2 + 12 X_3 - S_1 + a_1 = 96$$

$$X_1, X_2, X_3, S_1, S_2, a_1, a_2 \geq 0$$

### Phase I

$$\text{Minimize } Z = a_1 + a_2$$

Subject to



$$4X_1 + 8X_2 + 6X_3 - S_1 + a_1 = 64$$

$$3X_1 + 6X_2 + 12X_3 - S_1 + a_1 = 96$$

$$X_1, X_2, X_3, S_1, S_2, a_1, a_2 \geq 0$$

CB <sub>i</sub>	Basic Var	x <sub>1</sub>	x <sub>2</sub>	x <sub>3</sub>	s <sub>1</sub>	s <sub>2</sub>	a <sub>1</sub>	a <sub>2</sub>	Soln
1	a <sub>1</sub>	4	8	6	-1	0	1	0	64
1	a <sub>2</sub>	3	6	12	0	-1	0	1	96
	C <sub>j</sub> -Z <sub>j</sub>	-7	-14	-18	1	1	0	0	160
1	a <sub>1</sub>	5/2	5/2	0	-1/2	1	-1/2	16	14/5
0	x <sub>3</sub>	1/2	1/2	1	0	-1/2	0	1/2	8/16
	C <sub>j</sub> -Z <sub>j</sub>	-5/2	-5	0	1	-1/2	0	3/2	16
0	x <sub>2</sub>	1/2	1	0	-1/5	1/10	1/5	-1/10	16/5
0	x <sub>3</sub>	0	0	1	1/10	-2/15	-1/10	2/15	32/15
	C <sub>j</sub> -Z <sub>j</sub>	0	0	0	0	0	0	0	0

w=0

Phase I : x<sub>1</sub>=0, x<sub>2</sub>=16/5, x<sub>3</sub>=32/5

Phase II : From original z equation →

CB <sub>i</sub>	C <sub>j</sub>	x <sub>1</sub>	x <sub>2</sub>	x <sub>3</sub>	s <sub>1</sub>	s <sub>2</sub>	Soln
18	x <sub>2</sub>	1/2	1	0	-1/5	1/10	16/5
15	x <sub>3</sub>	0	0	1	1/10	-2/15	32/15
	Z <sub>j</sub>	9	18	15	-2/10	-1/5	768/15
	C <sub>j</sub> -Z <sub>j</sub>	3	0	0	2/10	1/5	

∴ Z (optimum) = 768/5