Game Theory

- For effective decision making.
- Decision making is classified into 3 categories:
    - Deterministic Situation:
        - Problem data representing the situation are constant.
        - They do not vary with respect to time or any other basis.
    - Probabilistic Situation:
        - Problem data representing the situation do not remain constant.
        - Vary due to chance.
        - Can be represented in the form of some probability distribution.
        - Later, these probability distributions become components of the decision making model of the situation.
    - Uncertainty Situation:
        - When the problem data are subjected to variation and it is not possible to represent them in the form of any probability distribution.
        - Game theory is an example of the uncertainty situation.

Terminologies of Game Theory:

Players: 2 players in a game.

Examples:-

- Any two companies competing for tenders.
- 2 countries planning for trade gains in a 3rd country.
- 2 persons bidding in a game.

Strategy: course of action taken by a player.

Examples:-

- Giving computer furniture free of cost.
- Giving 20% additional HW.
- Giving special price etc. while selling HW.

a) Pure Strategy.

b) Mixed Strategy.

m → no. of strategies of Player A.

n → no. of strategies of Player B.

p_i → Probability of selection of alternative *i* of player A (i = 1,2,3,.....,m).

q_j → Probability of selection of alternative *j* of player B (j = 1,2,3,.....,n).

$$\sum_{i=1}^{m} p_i = 1 \qquad and \qquad \sum_{j=1}^{n} q_j = 1$$

Pure Strategy:

- If a player selects a particular strategy with a probability of 1 it is pure strategy.
- Player is selecting that particular strategy alone ignoring his remaining strategies.

  $p_1+p_2+p_3+........+p_m = 1$

  $p_1+0+........0 = 1$

Mixed Strategy:

- Player selects more than one strategy.
- In this case probability of selection of the individual strategies will be less than one.

  Example:

  If $q_1 = 0.65$, $q_2 = 0$ and $q_3 = 0.35$ then $q_1 + q_2 + q_3 = 1$.

Pay Off Matrix [a_{ij}]:

- Each combination of the alternatives of the players A and B is associated with an outcome.

Pay Off Matrix of A.

$$
\begin{array}{c|ccccc|}
\text{(Player A)} \backslash \text{(Player B)} & 1 & 2 & \cdots \; j \; \cdots & & n \\
\hline
1 & \mathbf{a_{11}} & \mathbf{a_{12}} & \mathbf{a_{13}} \ldots\ldots \mathbf{a_{1j}} \ldots\ldots & & \mathbf{a_{jn}} \\
2 & \mathbf{a_{21}} & & & & . \\
3 & \mathbf{a_{31}} & & & & . \\
. & . & & & & . \\
. & . & & & & . \\
. & \mathbf{a_{i1}} & & & & . \\
 & . & & & & . \\
m & \mathbf{a_{m1}} & . & . \quad . & . & \mathbf{a_{mn}} \\
\hline
\end{array}
$$

Maximin Principle:

- Maximizes the minimum guaranteed gains of Player A.
- This is known as <u>*maximin value*</u>, and corresponding strategy is called <u>*maximin* strategy.</u>

Minimax Principle:

- Minimizes the maximum losses.
- This is known as <u>*minimax value*</u>, and corresponding strategy is called <u>*minimax* strategy.</u>

Saddle Point:

- If <u>*maximin value = minimax value*</u> game has saddle point.
- Intersecting cell corresponding to this value.
- Pure Strategy.

Value of the Game:

- If the game has a saddle point, then the value of the saddle point is the value of the game.

  Example:

Pay off Matrix of Company A

		(Company B)			
		1	2	3	Row minimum
	1	**20**	**15**	**22**	15
					35 (*maximin)
(Company A)	2	$\boxed{35}$	**45**	**40**	20
	3	**18**	**20**	**25**	
Column maximum		**35 (*minimax)**	45	40	

- Saddle Point (2,1)
- Value of game(A) = 35
- Value of game(B) = -35

- A should always select strategy 2 with a probability of 1, and it should select all other strategies with probability zero.
- B should always select strategy 1 with a probability of 1, and all other strategies with probability zero, for <u>optimal solution</u>.

<u>Game With Mixed Strategies</u>

- If a game has no saddle point, then the game is said to have mixed strategies.

		(B)		
		1	2	Oddments
(A)	1	**a**	**b**	| c-d |
	2	**c**	**d**	| a-b |
	Oddments	| b-d |	| a-b |	

Let p_1 and p_2 be the probabilities of selection of strategies 1 and 2 respectively for player A, and q_1 and q_2 for player B.

$$p_1 = \frac{|c-d|}{|a-b|+|c-d|}$$

$$p_2 = \frac{|a-b|}{|a-b|+|c-d|}$$

$$q_1 = \frac{|b-d|}{|a-c|+|b-d|}$$

$$p_1 = \frac{|a-c|}{|a-c|+|b-d|}$$

$$v = \frac{a|c-d|+c|a-b|}{|a-b|+|c-d|}$$
$$= \frac{b|c-d|+d|a-b|}{|a-b|+|c-d|}$$

$$= \frac{a|\, b-d\,| + b|a-c|}{|a-c| + |b-d|}$$

$$= \frac{c|\, b-d\,| + d|a-c|}{|a-c| + |b-d|}$$

Example:

(B)

		1	2	Row Min.
(A)	1	**6**	**9**	6 (maximin)
	2	**8**	**4**	4
	Column max	8 (minimax)	9	

- Here maximin is not equal to minimax.
- No saddle point.

(B)

		1	2	Oddments
(A)	1	**6**	**9**	8 - 4 = 4
	2	**8**	**4**	9 - 6 = 3
	Oddments	5	2	

$p_1 = 4 / 4+3 = 4 / 7$

$p_2 = 3 / 7$

$p_1 + p_2 = 4 / 7 + 3 / 4 = 1$

$q_1 = 5 / 7$

$q_2 = 2 / 7$

$v = 6*4+8*3 / 4+3 = 48 / 7$

Hence the strategies of: Player A is (4 / 7, 3 / 7)

and of Player B is (5 / 7, 2 / 7), Also Value of game = 48 / 7.

Dominance Property

1. **Dominance Property For Rows:**

   Clause (a): In the Pay Off matrix of player A, if all the entries in the row (X) are greater than or equal to the corresponding entries of row (Y), then row (Y) is dominated by row (X). In this case, row (Y) of Pay Off matrix can be deleted.

   Clause (b): In the Pay Off matrix of player A, if each of the sum of the entries of any two rows [row (X) + row (Y)] is greater than or equal to the corresponding entry of a third row (Z), then row (Z) is dominated by row (X) and row (Y). In this case, row (Z) of Pay Off matrix can be deleted.

2. **Dominance Property For columns:**

   Clause (a): In the Pay Off matrix of player A, if all the entries in the column (X) are greater than or equal to the corresponding entries of column (Y), then column (Y) is dominated by column (X). In this case, column (Y) of Pay Off matrix can be deleted.

   Clause (b): In the Pay Off matrix of player A, if each of the sum of the entries of any two columns [column (X) + column (Y)] is greater than or equal to the corresponding entry of a third column (Z), then column (Z) is dominated by column (X) and column (Y). In this case, column (Z) of Pay Off matrix can be deleted.

EXAMPLE:

Player A and B play a game in which each player has three coins (20p, 25p and 50p). Each of them selects a coin without the knowledge of the other person. If the sum of the values of the coins is even then A wins B's coin. If it is odd, then B wins A's coins.

1. Develop a Pay Off matrix w.r.t Player A.
2. Find the optimal strategies for the players.

SOLUTION:

(Player B)

		1 (20p)	2 (25p)	3 (50p)
	1 (20p)	**20**	**-20**	**50**
(Player A)	2 (25p)	**-25**	**25**	**-25**
	3 (50p)	**20**	**-50**	**50**

Step 1: Find saddle point.

No saddle point → apply mixed strategy.

Step 2: Check for dominance property.

Row 3 is dominated by Row 1.

(Player B)

(Player A)		1 (20p)	2 (25p)	3 (50p)
	1 (20p)	**20**	**-20**	**50**
	2 (25p)	**-25**	**25**	**-25**

No saddle point.

Column 3 is dominated by column 1.

(Player B)

(Player A)		1 (20p)	2 (25p)
	1 (20p)	**20**	**-20**
	2 (25p)	**-25**	**25**

(Player B)

(Player A)		1 (20p)	2 (25p)	Oddments
	1 (20p)	**20**	**-20**	50
	2 (25p)	**-25**	**25**	40
	Oddments	45	45	

$p_1 = 50 / 50+40 = 5 / 9$, $p_2 = 40 / 50+40 = 4 / 9$

$p_1 + p_2 = 1$

$q_1 = 45 / 45+45 = ½$, $q_2 = 45 / 45+45 = 1/2$

$v = 20*50 + (-25)*40 / 50+45 = 0$

Hence the strategies of: Player A is (5 / 9, 4 / 9, 0) and of Player B is (1 / 2, 1 / 2, 0)

Value of game = 0.