NUMERICAL AND STATISTICAL COMPUTING (MCA-202-CR)

Autumn Session

UNIT 1

Numerical analysis is the study of algorithms that uses, creates and implements algorithms for obtaining numerical solutions to problems involving continuous variables.

Numerical techniques are concerned with all aspects of the numerical solution of a problem, from the theoretical development and understanding of numerical methods to their practical implementation as reliable and efficient computer programs.

We study numerical analysis to

 - ➢ Get to know how fast errors can cause problems.
 - ➢ Find better algorithms that cause fewer errors.

Numerical techniques are further used to develop fast and accurate approximations to problems where exact solutions are difficult to find because of complexity.

APPLICATIONS OF NUMERICAL TECHNIQUES:

1. Traffic simulation
2. Environment simulation
3. Fluid dynamics.
4. Network analysis.
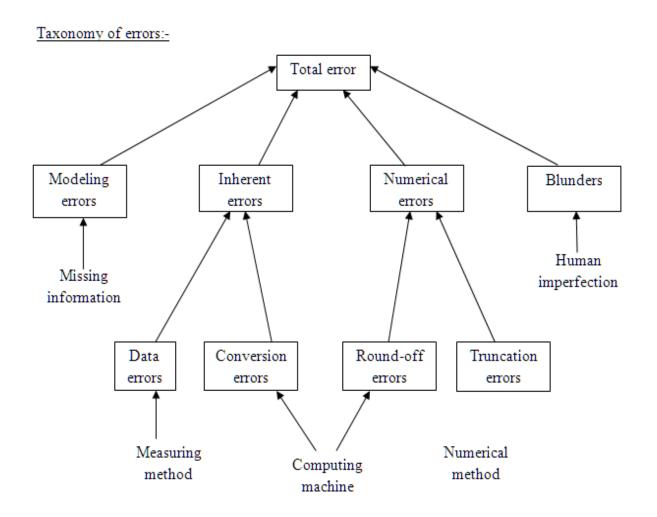5. Signal processing.

APPROXIMATION AND ERRORS:-

Approximation usually occurs when an exact form or an exact numerical number is unknown or difficult to obtain.

Error is defined as the difference between actual value and the approximate value obtained from the experimental observation or from numerical computation.

$$error = actual\ value - approximate\ value$$

Types Of Errors:

A number of different types of errors arise during the process of numerical computing. All these errors contribute to the total error in the total error in the final result.

Taxonomy of errors:-

1. **INHERENT ERRORS:** Inherent errors are those that are present in the data supplied to the model. Inherent errors contain two components: data errors and conversion errors.

   a. **Data errors:** Also known as empirical error, data errors arise when the data for a problem are obtained by some experimental means and are therefore of limited accuracy and precision. This may be due to some limitations in instrumentation and reading, and therefore may be unavoidable. For example, vernier caliper, screw guage etc can measure the quantity accurate to certain smallest value.

   b. **Conversion errors:** Also known as representation errors, conversion errors arise due to the limitation of the computer to store the data exactly. It is known that the floating point representation retains only a specified number of digits. The digits that are not retained constitute the round-off error.

2. **NUMERICAL ERRORS:** Numerical errors are also known as procedural errors. These errors are also introduced during the process of implementation of a numerical method. They come in two forms: round-off errors and truncation errors. The total

numerical error is the summation of these two errors. The total error can be reduced by devising suitable techniques for implementing the solution.

a. Round-off errors: Round-off errors occur when a fixed number of digits are used to represent exact numbers. Since the numbers are stored at every stage of computation, round-off error is introduced at the end of every arithmetic operation. Consequently, an individual round-off error could be very small, the cumulative effect of a series of computations can be very significant.

Rounding a number can be done in one of the 2 ways:

i. Chopping: In chopping, extra digits are dropped. This is called "truncating" the number. Suppose we are using a computer with a fixed word length of four digits. Then a number like 42.7893 will be stored as 42.78, and the digits 93 will be dropped.

ii. Symmetric round-off : In the symmetric round-off method, the last digit retained significant digit is "rounded up" by 1, if the discarded digit is larger or equal to 5; otherwise, the last digit retained is unchanged, e.g., the number 42.7893 would become 42.79 and the number 76.5432 would become 76.54.

b. Truncation errors: Truncation errors arise from using an approximation in place of an exact mathematical procedure. Typically, it is the error resulting from the truncation of the numerical process. Often some finite number of series is used to estimate the sum of an infinite series. Many of the iterative procedure used in numerical computing are infinite, and therefore, result in truncation errors. Truncation errors can be reduced by using a better numerical model which usually increases the number of arithmetic operations. But care should be exercised to see that the round-off error which is bound to increase due to increase in arithmetic operations does not offset the reduction in truncation error.

ACCURACY AND PRECISION:

Accuracy tells us how close a measured value is to the actual value. It is associated with the quality of data and numbers of errors present in the data set. Accuracy can be calculated using a single factor or measurement.

Precision tells us how close measured values are to each other. It often results in round off errors. To calculate precision, multiple measurements are required.

e.g.,   the distance between point A and B is 7.15. On measuring with different devices the distance appears as:

Data set1: 6.34, 6.31, 6.32

Data set2: 7.11, 7.19, 7.9

Data set 1 is more precise since they are close to each other and data set 2 is more accurate since they are close to the actual value.
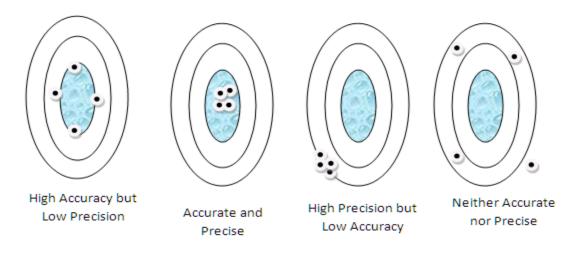

High Accuracy but Low Precision

Accurate and Precise

High Precision but Low Accuracy

Neither Accurate nor Precise

Fig: Target Analogy

A stopped clock may be accurate twice a day, but won't be precise.

MEASURES OF ACCURACY:

The 2 ways to measure accuracy are: absolute error and relative error.

1. ABSOLUTE ERROR: Absolute error is the measure of how far off measurement is from a true value or an indication of uncertainty of measurement.
$$Absolute\ error = Actual\ Value - Measured\ Value$$
   E.g. Length of book is 75mm +/- 1
   Absolute error = 1 mm
2. RELATIVE ERROR: Relative error expresses how large the absolute error is compared to the total size of object you are measuring.
$$Relative\ error = \frac{(Actual\ Error - Meaured\ Value)}{Actual\ Value}$$

   E.g. Drivers speedometer displays speed as 60 m/sec
   But Actual Value = 62 m/sec
   Relative Error = 2/60 =0.033=3.3%
   Relative error tells which has a smaller error.

Q1. Let X= 0.00458529. Find absolute error if x is truncated to 3 decimal digits.
Sol. $X = 0.00458529 = 0.458529*10^{-2}$
                   $X_a = 0.458*10^{-2}$

$$\text{Absolute Error} = (0.458529*10^{-2}) - (0.458*10^{-2})$$
$$= 0.529*10^{-5}$$

Q2. Let X= 0.005998. Find relative error if x is truncated to 3 decimal digits.

Sol.
$$X = 0.005998 = 0.5998*10^{-2}$$
$$X_a = 0.599*10^{-2}$$
$$\text{Relative Error} = (0.5998*10^{-2} - 0.599*10^{-2}) / (0.5998*10^{-2})$$
$$= 0.00133$$
$$= 0.133*10^{-2}$$

Q3. Let X= 0.00458529. Find absolute error if x is rounded-off to 3 decimal digits.

Sol.
$$X = 0.00458529 = 0.458529*10^{-2}$$
$$X_a = 0.459*10^{-2}$$
$$\text{Absolute Error} = (0.458529*10^{-2}) - (0.459*10^{-2})$$
$$= 0.471*10^{-5}$$

ERROR PROPAGATION

Numerical solutions involve a series of computational steps. Therefore, it is necessary to understand the way the error propagates with progressive computations.

Propagation of error (or propagation of uncertainty) is the effect of variables, uncertainties (or errors) on the uncertainty of a function based on them.

TYPES OF EQUATIONS

There are two types of equations: linear equations and non linear equations.

1. <u>LINEAR EQUATIONS</u>: Linear equations is a polynomial of degree one.
2. <u>NON-LINEAR EQUATIONS</u>: The non-linear equations fall in following categories:
    a. Polynomial: Polynomials are expressions of more than two algebraic terms. The general form of polynomial is:
    $$a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_2 x^2 + a_1 x + a_0 = 0, \quad \text{where } a_n != 0$$
    It is n^{th} degree polynomial in x and has n roots. These roots may be :
    > Real and different
    > Real and repeated
    > Complex
    b. Transcendental: A non-polynomial equation is called transcendental equation.
    e.g. $Xe^x - X\sin X = 0, \quad 2^x - x - 3 = 0$

A transcendental equation may have finite/ infinite number of real roots or may not have any real root at all.

METHODS OF FINDING SOLUTIONS OF NON-LINEAR EQUATIONS

The two kinds of methods to obtain solutions of non-linear equations are:

1. Direct Methods: Direct methods give the roots of non-linear equations in a finite number of steps. They are capable of giving all the roots at the same time.
2. Iterative Methods: Also known as '*trial and error method',* iterative methods are based on the idea of successive approximations. They start with one or more initial approximations to the root and obtain a sequence of approximations by repeating a fixed sequence of steps till the solution with reasonable accuracy is obtained. They generally give one root at a time. They are cumbersome and time consuming.

ALGORITHMS TO COMPUTE ROOTS OF EQUATION:

1. Choosing Initial Approximation: The roots of the equation are the points where the curve representing function f(x) intersects the x-axis. Therefore, the interval where the function changes its sign is taken as initial approximation.
   The best way to find a root is either to plot the function f(x) or to tabulate it.
   The roots of equation are the points where the curve representing the function f(x) intersects the x- axis. Therefore, any point in the interval where the function changes its sign can be taken as initial approximation.
2. Finding the Root:
   If f(x1)*f(x3) is less than x2=x3
   If f(x1)*f(x3) is greater than x1=x3
   If f(x1)*f(x3) is equal to 0, then the root is found.

3. When to terminate: Iterative procedure is continued till the required degree of accuracy is achieved.
   Termination Criteria1 (TC1): if $|(x_{i+1})| \leq$ epsilon
   Termination Criteria2 (TC2): if $|(x_{i+1} - x_i)| \leq$ epsilon
   Termination Criteria3 (TC3): if $|((x_{i+1} - x_i) / x_{i+1})| \leq$ epsilon
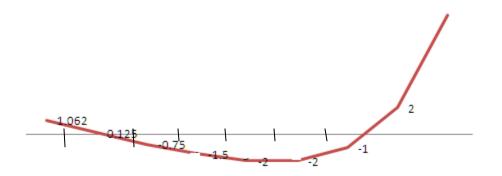
Question: Find the root of the equation 2^x-x-3 = 0 graphically.
Solution: F (-4) = 2^{-4}-(-4)-3 = 1.062
      F (-3) = 2^{-3}-(-3)-3 = 0.125
      F (-2) = 2^{-2}-(-2)-3 = -0.75
      F (-1) = 2^{-1}-(-1)-3 = -1.5
      F (0) = 2^0-0-3 = -2
      F (1) = 2^1-1-3 = -2
      F (2) = 2^2-2-3 = -1
      F (3) = 2^3-3-3 = 2

$$F(4) = 2^4\text{-}4\text{-}3 = 9$$

Tabulated data:

F(x)	-4	-3	-2	-1	0	1	2	3	4
X	1.062	0.125	-0.75	-1.5	-2	-2	-1	2	9

Therefore root lies between -2 and -3, or between 2 and 3.

BISECTION METHOD

Also known as BOLZANO METHOD, bisection method is the simplest iterative method.

ALGORITHM:

1. Initial Condition: To apply bisection method, both of these conditions should be true :
    a. It has to be a continuous value.
    b. Sign of f(x1) should not be same as that of f(x2).
2. In each iteration

    It starts with two initial approximations, x1 and x2 such that f(x1)*f(x2) <0. The next point x3 is calculated as:
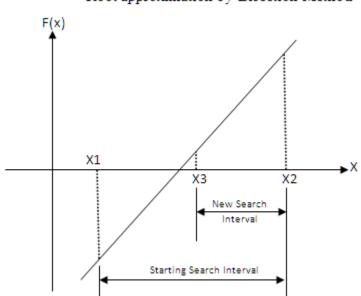 $$x3 = (x1+x2)/2$$
    Calculate f(x3)

    If f(x1) and f(x3) are of opposite sign, replace x2 by x3.
    If f(x1) and f(x3) are of same sign, replace x1 by x3.
    If f(x3) is equal to 0, then x3 is the root.

3. When to terminate:
    a. If f(x1)*f(x2) = 0, then root is found.
    b. If |x1-x2|<= 0 or |x1-x2|<= epsilon, then terminate the algorithm.

c. If $(x3^{new} - x3^{old}) / x3^{new} <= 0$ or epsilon, then terminate the algorithm.

Root approximation by Bisection Method

ADVANTAGE:

    a. It is always convergent.
    b. Error can be controlled, since error gets halved in each iteration.

DISADVANTAGE:

    a. Convergence is usually slow (although it is guaranteed).
    b. Choosing guess close to the root may result in needing many iterations to converge.

Example

Q: Find the root of the equation correct to three decimal digits using Bisection Method.

$$X^3 - 2X - 5 = 0$$

Sol: Finding the initial approximation:

$F(0) = 0^3 - 2(0) - 5 = -5$
$F(1) = 1^3 - 2(1) - 5 = -6$
$F(2) = 2^3 - 2(2) - 5 = -1$
$F(3) = 3^3 - 2(3) - 5 = 16$

Since f (2) is negative and f (3) is positive, therefore root lies between 2 and 3.

ITERATION 1: X1=2,   X2 = 3
        X3= (X1+X2)/2 = (2+3)/2 =2.5
        F (2.5) = $(2.5)^3$ – 2(2.5) - 5 = 5.625
Therefore, the first approximation to root is 2.5.
Since sign of f(x1) and f(x3) is different, therefore replace X2 by X3.

ITERATION 2: X1=2,   X2 = 2.5
        X3=(X1+X2)/2 = (2+3)/2 =2.25
        F (2.25) = $(2.25)^3$ – 2(2.25) - 5 = 1.89
Therefore, the second approximation to root is 2.25.
Since sign of f(x1) and f(x3) is different, therefore replace X2 by X3.

ITERATION 3: X1=2,   X2 = 2.25
        X3=(X1+X2)/2 = (2+2.25)/2 =2.125
        F (2.125) = $(2.125)^3$ – 2(2.125) - 5 = 0.345
Therefore, the third approximation to root is 2.125.
Since sign of f(x1) and f(x3) is different, therefore replace X2 by X3.

ITERATION 4: X1=2,   X2 = 2.125
        X3= (X1+X2)/2 = (2+2.125)/2 =2.062
        F (2.062) = $(2.062)^3$ – 2(2.062) - 5 = -0.357
Therefore, the fourth approximation to root is 2.062.
Since sign of f(x1) and f(x3) is same, therefore replace X1 by X3.

ITERATION 5: X1=2.062,   X2 = 2.125
        X3= (X1+X2)/2 = (2.062+2.125)/2 =2.093
        F (2.093) = $(2.093)^3$ – 2(2.093) - 5 = -0.018
Therefore, the fifth approximation to root is 2.093.
Since sign of f(x1) and f(x3) is same, therefore replace X1 by X3.

ITERATION 6: X1=2.093,   X2 = 2.125
        X3=(X1+X2)/2 = (2.093+2.125)/2 =2.109
        F (2.109) = $(2.109)^3$ – 2(2.109) - 5 = 0.162
Therefore, the sixth approximation to root is 2.109.
Since sign of f(x1) and f(x3) is different, therefore replace X2 by X3.

ITERATION 7: X1=2.093,   X2 = 2.109
        X3=(X1+X2)/2 = (2.093+2.109)/2 =2.101

$F (2.101) = (2.101)^3 – 2(2.101) - 5 = 0.072$

Therefore, the seventh approximation to root is 2.101.

Since sign of f(x1) and f(x3) is different, therefore replace X2 by X3.

ITERATION 8: X1=2.093,   X2 = 2.101

       X3=(X1+X2)/2 = (2.093+2.101)/2 =2.097

       $F (2.097) = (2.097)^3 – 2(2.097) - 5 = 0.027$

Therefore, the eighth approximation to root is 2.097.

Since sign of f(x1) and f(x3) is different, therefore replace X2 by X3.

ITERATION 9: X1=2.093,   X2 = 2.097

       X3= (X1+X2)/2 = (2.093+2.097)/2 =2.095

       $F (2.095) = (2.095)^3 – 2(2.095) - 5 = 0.005$

Therefore, the ninth approximation to root is 2.095.

Since sign of f(x1) and f(x3) is different, therefore replace X2 by X3.

ITERATION 10: X1=2.093,   X2 = 2.095

       X3=(X1+X2)/2 = (2.093+2.095)/2 =2.094

       $F (2.094) = (2.094)^3 – 2(2.094) - 5 = -0.007$

Therefore, the tenth approximation to root is 2.094.

Since sign of f(x1) and f(x3) is same, therefore replace X1 by X3.

ITERATION 11: X1=2.094,   X2 = 2.095

       X3=(X1+X2)/2 = (2.094+2.095)/2 =2.094

       $F (2.094) = (2.094)^3 – 2(2.094) - 5 = 0.027$

Therefore, the eighth approximation to root is 2.094

Since we want the solution correct to three decimal places, the iterative procedure will terminate as soon as the successive iterations produce no change at first three decimal positions.
Observing the last two iterations, they don't produce any change at the first three decimal places.
Therefore we take x=2.094 as the desired solution correct to three decimal positions.

ALGORITHMS:

To find the root of f(x) = 0 within a prescribed tolerance say Epsilon. Given values of x1 and x2 such that f(x1)*f(x2) < 0, the variable x3 is used to store mid-point of the interval.

BEGIN

Read: x1,x2   //input value for x1 and x2

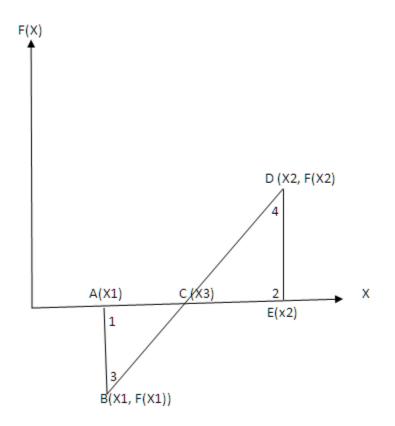Read: Epsilon   //input prescribed tolerance

Do

{

Set x3 = (x1+x2)/2   //Compute mid-point
If (f(x1)*f(x2))<0 then
      Set x2=x3
Else
      Set x1=x3
End if
} while ($|x1-x2|/x2$ >epsilon)
Write: x3,"is the approximate root"
End

FALSE – POSITION / REGULA FALSI / METHOD OF LINEAR INTERPOLATION

Also known as 'REGULA FASLI' or 'METHOD OF LINEAR INTERPOLATION', false position method is similar to bisection method but is faster than it.

DERIVATION:

Plot points x1 and x2 (initial approximations). Join the points (x1, f(x1)) and (x2,f(x2)) by a straight line. The point where this line intersects the x-axis(x3) is the next approximation to the root.

We know that , if two angles of a triangle have measure equal to the measures of two angles of another triangle, then the triangles are similar i.e., corresponding sides of similar polygon are in proportion.

Here, $<1 = <2$ (both 90°)
      $<3 = <4$ (alternate angles)
      Therefore, triangle ABC ~ triangle EDC.

Also,

$$= \frac{AB}{DE} = \frac{AC}{CE}$$
$$= \frac{F\,(X1)}{F\,(X2)} = \frac{X3-X1}{X2-X3}$$
$$= \frac{F\,(X1)}{F\,(X2)} = \frac{-X3+X1}{X2-X3}$$
$$= F\,(X1) * (X2 - X3) = (F\,(X2) * X1) - (F\,(X2) * X3)$$
$$= (F\,(X1) * X2) - F(X1) * X3 = (F\,(X2) * X1) - (F\,(X2) * X3)$$
$$= (F\,(X2) * X3) - F(X1) * X3 = (F\,(X2) * X1) - (F\,(X2) * X2$$
$$= X3(\,F(X2) - F(X1)) = (F\,(X2) * X1) - (F\,(X1) * X2$$
$$= X3 = \frac{(X1*F\,(X2))-(X2*F(X1))}{F(X2)-F(X1)}$$

Example

Q: Find the root of the equation correct to three decimal digits using False Position Method.

$$X^3 - 2X - 5 = 0$$

Sol: Finding the initial approximation:

   F (0) = $0^3 - 2(0) - 5 = -5$
   F (1) = $1^3 - 2(1) - 5 = -6$
   F (2) = $2^3 - 2(2) - 5 = -1$
   F (3) = $3^3 - 2(3) - 5 = 16$

Since f (2) is negative and f (3) is positive, therefore root lies between 2 and 3.

<u>ITERATION 1</u>: X1=2,   X2 = 3

$$X3 = \frac{(X1*F\,(X2))-(X2*F(X1))}{F(X2)-F(X1)}$$

$$X3 = \frac{(2*16)-(3*-1)}{16-(-1)}$$

$$X3 = 2.058$$

F (2.058) = $(2.058)^3$ − 2(2.058) - 5 = -0.4.

Therefore, the first approximation to root is 2.058.

Since sign of f(x1) and f(x3) is same, therefore replace X1 by X3.

<u>ITERATION 2</u>: X1=2.058,   X2 = 3

$$X3 = \frac{(X1*F\,(X2))-(X2*F(X1))}{F(X2)-F(X1)}$$

$$X3 = \frac{(2.058*16)-(3*-0.4)}{16-(-0.4)}$$

$$X3 = 2.056$$

F (2.056) = $(2.056)^3$ − 2(2.056) - 5 = -0.422.

Therefore, the second approximation to root is 2.056.

Since sign of f(x1) and f(x3) is same, therefore replace X1 by X3.

<u>ITERATION 3</u>: X1=2.056,   X2 = 3

$$X3 = \frac{(X1*F\,(X2))-(X2*F(X1))}{F(X2)-F(X1)}$$

$$X3 = \frac{(2.056*16)-(3*-0.422)}{16-(-0.422)}$$

$$X3 = 2.08$$

F (2.08) = $(2.08)^3$ − 2(2.08) - 5 = -0.167.

Therefore, the third approximation to root is 2.08.

Since sign of f(x1) and f(x3) is same, therefore replace X1 by X3.

<u>ITERATION 4</u>: X1=2.08,   X2 = 3

$$X3 = \frac{(X1*F\,(X2))-(X2*F(X1))}{F(X2)-F(X1)}$$

$$X3 = \frac{(2.08*16)-(3*-0.162)}{16-(-0.162)}$$

$$X3 = 2.089$$

F (2.058) = $(2.058)^3$ – 2(2.058) - 5 = -0.062.

Therefore, the fourth approximation to root is 2.089.

Since sign of f(x1) and f(x3) is same, therefore replace X1 by X3.

ITERATION 5: X1=2.089,   X2 = 3

$$X3 = \frac{(X1*F(X2))-(X2*F(X1))}{F(X2)-F(X1)}$$

$$X3 = \frac{(2.089*16)-(3*-0.062)}{16+0.062}$$

$$X3 = 2.092$$

F (2.092) = $(2.092)^3$ – 2(2.092) - 5 = -0.029.

Therefore, the fifth approximation to root is 2.092.

Since sign of f(x1) and f(x3) is same, therefore replace X1 by X3.

ITERATION 6: X1=2.092,   X2 = 3

$$X3 = \frac{(X1*F(X2))-(X2*F(X1))}{F(X2)-F(X1)}$$

$$X3 = \frac{(2.092*16)-(3*-0.029)}{16+0.029}$$

$$X3 = 2.093$$

F (2.093) = $(2.093)^3$ – 2(2.093) - 5 = -0.018.

Therefore, the sixth approximation to root is 2.093.

Since sign of f(x1) and f(x3) is same, therefore replace X1 by X3.

ITERATION 7: X1=2.093,   X2 = 3

$$X3 = \frac{(X1*F(X2))-(X2*F(X1))}{F(X2)-F(X1)}$$

$$X3 = \frac{(2.093*16)-(3*-0.018)}{16+0.018}$$

$$X3 = 2.094$$

F (2.094) = $(2.094)^3$ – 2(2.094) - 5 = -0.007.

Therefore, the seventh approximation to root is 2.094.
Since sign of f(x1) and f(x3) is same, therefore replace X1 by X3.

ITERATION 8: X1=2.094,   X2 = 3

$$X3 = \frac{(X1*F(X2))-(X2*F(X1))}{F(X2)-F(X1)}$$

$$X3 = \frac{(2.094*16)-(3*-0.007)}{16+0.007}$$

$$X3 = 2.094.$$

Therefore, the eighth approximation to root is 2.094.
Since we want the solution correct to three decimal places, the iterative procedure will terminate as soon as the successive iterations produce no change at first three decimal positions.
Observing the last two iterations, they don't produce any change at the first three decimal places.
Therefore we take x=2.094 as the desired solution correct to three decimal positions

SECANT METHOD

Secant method is used to find root of non-linear equation of form f(x) = 0. It is similar to false position with only one difference. In the secant method, two most recent approximations to the root are used to find the next approximation. Further, it is not necessary for the initial approximations to enclose the root. The secant method is much faster than false position method, but convergence is not always assured and therefore we put a limit on the number of iterations permitted.

Let x1 and x2 be two initial approximations to the root. Join the points (x1, f(x1)) and (x2, f(x2)) by a straight line. The point where this line intersects the x-axis gives the next approximation to the root and is named as x3. Now x2 and x3 are taken as starting point for next iteration. The line is drawn joining the points (x2, f(x2)) and (x3,f(x3)). Let this intersection point be x4. The iterative procedure terminates when the relative error in two successive approximations is less than or equal to the prescribed tolerance.

$$X_{i+1} = (\ (X_{i-1}*f(x_i)) - (X_i*f(x_{i-1}))\ ) / (f(x_i) - f(x_{i-1}))$$

Example
Q: Find the root of the equation correct to three decimal digits using Secant Method.

$$X^3 - 2X - 5 = 0$$

Sol: Finding the initial approximation:

$F(0) = 0^3 - 2(0) - 5 = -5$
$F(1) = 1^3 - 2(1) - 5 = -6$
$F(2) = 2^3 - 2(2) - 5 = -1$
$F(3) = 3^3 - 2(3) - 5 = 16$

Since f (2) is negative and f (3) is positive, therefore root lies between 2 and 3.

ITERATION 1: X1=2,   X2 = 3

$$X3 = \frac{(X1*F(X2))-(X2*F(X1))}{F(X2)-F(X1)}$$

$$X3 = \frac{(2*16)-(3*-1)}{16-(-1)}$$

$$X3 = 2.058$$

$F(2.058) = (2.058)^3 - 2(2.058) - 5 = -0.4.$

Therefore, the first approximation to root is 2.058.

ITERATION 2: Since we take last two approximations we have
            X1=3,   X2 = 2.058

$$X3 = \frac{(X1*F(X2))-(X2*F(X1))}{F(X2)-F(X1)}$$

$$X3 = \frac{(3*-0.4)-(2.058*16)}{-0.4-16}$$

$$X3 = 2.08$$

$F(2.08) = (2.08)^3 - 2(2.08) - 5 = -0.162.$

Therefore, the second approximation to root is 2.08.

ITERATION 3: Since we take last two approximations we have
            X1=2.058,   X2 = 2.08

$$X3 = \frac{(X1*F(X2))-(X2*F(X1))}{F(X2)-F(X1)}$$

$$X3 = \frac{(2.058*-0.162)-(2.08*-0.4)}{0.162+0.4}$$

$$X3 = 2.096$$

$F(2.096) = (2.096)^3 - 2(2.096) - 5 = 0.016.$

Therefore, the third approximation to root is 0.016.

ITERATION 4: Since we take last two approximations we have

X1=2.08,   X2 = 2.096

$$X3 = \frac{(X1 * F\ (X2)) - (X2 * F(X1))}{F(X2) - F(X1)}$$

$$X3 = \frac{(2.08 * 0.016) - (2.096 * -0.162)}{0.016 - (-0.162)}$$

X3 = 2.089

F (2.089) = $(2.089)^3$ – 2(2.089) -5= -0.062.

Therefore, the fourth approximation to root is 2.089.

ITERATION 5: Since we take last two approximations we have

X1=2.096,   X2 =2.089

$$X3 = \frac{(X1 * F\ (X2)) - (X2 * F(X1))}{F(X2) - F(X1)}$$

$$X3 = \frac{(2.096 * -0.062) - (2.089 * 0.016)}{-0.062 - 0.016}$$

X3 = 2.076

F (2.076) = $(2.076)^3$ – 2(2.076) -5= -0.205.

Therefore, the fifth approximation to root is 2.076.

ITERATION 6: Since we take last two approximations we have

X1=2.089,   X2 =2.076

$$X3 = \frac{(X1 * F\ (X2)) - (X2 * F(X1))}{F(X2) - F(X1)}$$

$$X3 = \frac{(2.089 * -0.205) - (2.076 * -0.062)}{-0.205 - (-0.062)}$$

X3 = 2.097

F (2.097) = $(2.097)^3$ – 2(2.097) -5= 0.027.

Therefore, the sixth approximation to root is 2.097.

ITERATION 7: Since we take last two approximations we have

X1=2.076,   X2 = 2.097

$$X3 = \frac{(X1 * F\ (X2)) - (X2 * F(X1))}{F(X2) - F(X1)}$$

$$X3 = \frac{(2.076 * 0.027) - (2.097 * -0.205)}{0.027 - (-0.205)}$$

X3 = 2.073

F (2.073) = $(2.073)^3$ – 2(2.073) -5= -0.238.

Therefore, the seventh approximation to root is 2.073.

ITERATION 8: Since we take last two approximations we have

$X1=2.097,$ $\qquad$ $X2 = 2.073$

$$X3 = \frac{(X1*F\ (X2))-(X2*F(X1))}{F(X2)-F(X1)}$$

$$X3 = \frac{(2.097*-0.238)-(2.073*0.027)}{-0.238-0.027}$$

$$X3 = 2.090$$

$F\ (2.090) = (2.090)^3 - 2(2.090) -5= -0.051.$

Therefore, the eighth approximation to root is 2.090.

ITERATION 9: Since we take last two approximations we have

$X1=2.073,$ $\qquad$ $X2 = 2.090$

$$X3 = \frac{(X1*F\ (X2))-(X2*F(X1))}{F(X2)-F(X1)}$$

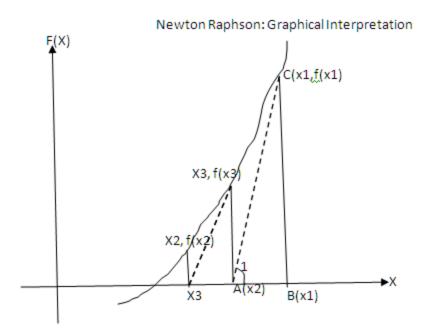$$X3 = \frac{(2.073*-0.051)-(2.090*-0.238)}{-0.051-(-0.238)}$$

$$X3 = 2.090$$

Therefore, the ninth approximation to root is 2.090.

Since we want the solution correct to three decimal places, the iterative procedure will terminate as soon as the successive iterations produce no change at first three decimal positions.
Observing the last two iterations, they don't produce any change at the first three decimal places.
Therefore we take x=2.090 as the desired solution correct to three decimal positions.

NEWTON RAPHSON METHOD

The Newton- Raphson method, also known as 'Newton's method of tangents', is one of the fastest iterative methods. This method begins with a single initial approximation.

GEOMETRIC INTERPRETATION:

Newton Raphson: Graphical Interpretation

Tan1= perpendicular/ base = (B-C) / (C-A) = F(x1)/(x1-x2)

Also, tan1 = $f'(x1)$

$\Rightarrow f'(x1) = f(x1)/(x1-x2)$

$\Rightarrow x2 = x1 - f(x1) / f'(x1)$

If the proper choice of the initial approximation is not made, the method may diverge. This causes the iterative procedure to fall in endless loop. Therefore, to put an end to such conditions we put a limit on the maximum number of iterations permitted.

ANALYTICAL DERIVATION:

Taylor series is a representation of a function as an infinite sum of terms that are calculated from the values of functions derivative at a single point.

If $f(x_i) = 0$ Then, $f(x_{i+h}) = 0$

Where x_i = initial approximation

x_{i+h} = next approximation

Using Taylors series,

$f(x_{i+h}) = f(x_i) + hf'(x_i) + h^2 f''(x_i)/2! + - - - - - + h^n f^n(x_i)/n!$

Where $f'(x_i)$ = first derivative

$f''(x_i)$ = second derivative and so on..

We are looking for a point where $f(x_i+h) = 0$

$f(x_i) + hf'(x_i) + h^2 f''(x_i)/2! + - - - - - + h^n f^n(x_i)/n! = 0$

Limiting and taking only two terms i.e. neglecting higher order terms.

$$f(x_i) + hf'(x_i) \sim = 0$$

Also,

h= x2-x1

$$f(x_i) + (f'(C) * (x2\text{-}x1)) = 0$$
$$f'(x_i)*(x2\text{-}x1) = \text{-}f(x_i)$$
$$x2 = x1 - f(x1)/ f'(x_i)$$

Example

Q: Find the root of the equation correct to three decimal digits using Secant Method.

$$X^3 - 2X - 5 = 0$$

Sol: $F(x) = X^3 - 2X - 5$

Therefore,

$$F'(x) = 3X^2 - 2$$

Finding the initial approximation:

$$F(0) = 0^3 - 2(0) - 5 = -5$$
$$F(1) = 1^3 - 2(1) - 5 = -6$$
$$F(2) = 2^3 - 2(2) - 5 = -1$$
$$F(3) = 3^3 - 2(3) - 5 = 16$$

Since f (2) is negative and f (3) is positive, therefore root lies between 2 and 3.

ITERATION 1: We start with a single approximation, X0=3

$$F(x) = X^3 - 2X - 5 = 3^3 - 2(3) - 5 = 16$$
$$F'(x) = 3X^2 - 2 = 3(3)^2 - 2 = 25$$
$$X2 = X1 - \frac{F(X1)}{F'(X1)}$$
$$X2 = 3 - \frac{16}{25}$$

X2 = 2.36

Therefore, the first approximation to root is 2.36.

ITERATION 2: Taking X1 = 2.36

$$F(x) = X^3 - 2X - 5 = 2.36^3 - 2(2.36) - 5 = 3.424$$
$$F'(x) = 3X^2 - 2 = 3(2.36)^2 - 2 = 14.7088$$
$$X2 = X1 - \frac{F(X1)}{F'(X1)}$$

$$X2 = 2.36 - \frac{3.424}{14.7088}$$

X2 = 2.13

Therefore, the second approximation to root is 2.13.

ITERATION 3: Taking X1 = 2.13

$F(x) = X^3 - 2X - 5 = 2.13^3 - 2(2.13) - 5 = 0.403$

$F'(x) = 3 X^2 - 2 = 3(2.13)^2 - 2 = 11.6107$

$$X2 = X1 - \frac{F(X1)}{F'(X1)}$$

$$X2 = 2.13 - \frac{0.403}{11.6107}$$

X2 = 2.096

Therefore, the third approximation to root is 2.096

ITERATION 4: Taking X1 = 2.096

$F(x) = X^3 - 2X - 5 = 2.096^3 - 2(2.096) - 5 = 0.016$

$F'(x) = 3 X^2 - 2 = 3(2.096)^2 - 2 = 11.179$

$$X2 = X1 - \frac{F(X1)}{F'(X1)}$$

$$X2 = 2.096 - \frac{0.016}{11.179}$$

X2 = 2.095

Therefore, the fourth approximation to root is 2.095

ITERATION 5: Taking X1 = 2.095

$F(x) = X^3 - 2X - 5 = 2.095^3 - 2(2.095) - 5 = 0.005$

$F'(x) = 3 X^2 - 2 = 3(2.095)^2 - 2 = 11.167$

$$X2 = X1 - \frac{F(X1)}{F'(X1)}$$

$$X2 = 2.095 - \frac{0.005}{11.167}$$

X2 = 2.094

Therefore, the fifth approximation to root is 2.094

ITERATION 6: Taking X1 = 2.094

$F(x) = X^3 - 2X - 5 = 2.094^3 - 2(2.094) - 5 = -0.007$

$$F^{\bullet}(x) = 3\ X^2 - 2 = 3(2.094)^2 - 2 = 11.154$$

$$X2 = X1 - \frac{F(X1)}{F'(X1)}$$

$$X2 = 2.094 - \frac{-0.007}{11.154}$$

$$X2 = 2.094$$

Since we want the solution correct to three decimal places, the iterative procedure will terminate as soon as the successive iterations produce no change at first three decimal positions.
Observing the last two iterations, they don't produce any change at the first three decimal places.
Therefore we take x=2.090 as the desired solution correct to three decimal positions

EXERCISE

Q: Solve the following using Bisection method, False Position, Secant Method and Newton Raphson

    a. $F(x) = x^3 - 5x + 1 = 0$
    b. $F(x) = x^3 = 20$
    c. $F(x) = \cos x - 3x + 1 = 0$
    d. $F(x) = \cos x - xe^x = 0$
    e. $F(x) = \sqrt{12}$
       Find the approximate values up to 3 decimal places.

Q: List the differences between Bisection method, False Position, Secant Method and Newton Raphson Method .

Q: Programmatic implementation of Bisection method,False Position, Secant Method and Newton Raphson in C or C++ language.