NUMERICAL AND STATISTICAL COMPUTING (MCA-202-CR)

Autumn Session

UNIT 2

TYPES OF EQUATIONS

There are two types of equations: linear equations and non linear equations.

1. LINEAR EQUATIONS: Linear equations is a polynomial of degree one.
2. NON-LINEAR EQUATIONS: The non-linear equations fall in following categories:
    a. Polynomial: Polynomials are expressions of more than two algebraic terms. The general form of polynomial is:

$$a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \dots + a_2 x^2 + a_1 x + a_0 = 0, \quad \text{where } a_n != 0$$

    It is n^{th} degree polynomial in x and has n roots. These roots may be :

    Real and different
    Real and repeated
    Complex

    b. Transcendental: A non-polynomial equation is called transcendental equation.

    e.g. $Xe^x - X\sin X = 0$, $2^x - x - 3 = 0$

A transcendental equation may have finite/ infinite number of real roots or may not have any real root at all.

METHODS OF FINDING SOLUTIONS OF NON-HOMOGENOUS SYSTEM OF LINEAR EQUATIONS

The two kinds of methods to obtain solutions of non-linear equations are:

a. Direct Methods: Also known as 'reduction method', direct methods are capable of giving all the roots at the same time. E.g.
    a. Gauss Elimination Method.
    b. Gauss Jordan Method.
    c. Crout's Method (LU Decomposition).
    d. Methods Inversion Method.
b. Iterative Methods:. They start with one or more initial approximations to the root and obtain a sequence of approximations by repeating a fixed sequence of steps till the solution with reasonable accuracy is obtained. E.g.
    a. Gauss - Seidel Method.
    b. Jacobi's Method.

ALGORITHMS TO SOLVE LINEAR ALGEBRAIC EQUATIONS:

1. GAUSS ELIMINATION

This is one of the most widely used methods. It is a systematic process of eliminating unknowns from the linear equations. This method is divided into two parts:

a. Triangularization
b. Back substitution

The steps of '*n*' equations in '*n*' unknowns are reduced to an equivalent triangular system (an equivalent system is a system having identical solution) of equation of type

$$A_{11} X_1 + A_{12} X_2 + A_{13} X_3 + \text{--------} + A_{1n}X_n = \mathbf{B}_1$$
$$A_{22} X_2 + A_{23} X_3 + \text{--------} + A_{2n}X_n = \mathbf{B}_2$$
$$A_{33} X_3 + \text{--------} + A_{3n}X_n = \mathbf{B}_3$$
$$A_{nn} X_n = \mathbf{B}_n$$

Using back substitution procedure we can solve this new equivalent system of equations.

Steps to Solve An Equation Using Gauss Elimination:

PART I: TRIANGULARIZATION:

Step 1: Eliminate x_1 from 2^{nd} equation onwards. This is done through:

a. Subtract from the second equation a_{21}/a_{11} times the first equation. This results in

$$\left[a21 - \frac{a21}{a11} * a11\right] x1 + \left[a22 - \frac{a21}{a11} * a12\right] x2 + \cdots + \left[a2n - \frac{a21}{a11} * a1n\right] xn = b2 - \frac{a21}{a11} b1$$

b. Similarly, subtract from the third equation a_{21}/a_{11} times the first equation. This will result in

$a_{32}x_2 + a_{33}x_3 + \text{- - -} + a_{3n}x_n = b_3$

c. Repeat this process till n^{th} equation is operated, and we get a new system of equation as:

$$A_{11} X_1 + A_{12} X_2 + A_{13} X_3 + \text{--------} + A_{1n}X_n = \mathbf{B}_1$$
$$A_{22} X_2 + A_{23} X_3 + \text{--------} + A_{2n}X_n = \mathbf{B}_2$$
$$A_{32} X_2 + A_{33} X_3 + \text{--------} + A_{3n}X_n = \mathbf{B}_3$$
$$A_{n2} X_2 + A_{n3} X_3 + \text{--------} + A_{nn}X_n = \mathbf{B}_n$$

Step 2: Eliminate x_2 from 3^{rd} equation onwards. This is done through:

a. Subtract from the third equation a_{32}/a_{22} times the second equation.
b. Similarly, subtract from the fourth equation a_{42}/a_{22} times the second equation.
c. Repeat this process till n^{th} equation is operated, and we get a new system of equation as:

$$A_{11} X_1 + A_{12} X_2 + A_{13} X_3 + \text{--------} + A_{1n}X_n = \mathbf{B}_1$$

A22 X2 + A23 X3 + -------- + A2nXn = **B**2

$$A_{33}\ X_3 + \text{--------} + A_{3n}X_n = \mathbf{B}_3$$
$$A_{n3}\ X_3 + \text{--------} + A_{nn}X_n = \mathbf{B}_n$$

The process will continue till the last equation contains only one unknown, namely x_n..The final form of equation will look like:

$$A_{11}\ X_1 + A_{12}\ X_2 + A_{13}\ X_3 + \text{--------} + A_{1n}X_n = \mathbf{B}_1$$
$$A_{22}\ X_2 + A_{23}\ X_3 + \text{--------} + A_{2n}X_n = \mathbf{B}_2$$
$$A_{33}\ X_3 + \text{--------} + A_{3n}X_n = \mathbf{B}_3$$
$$A_{nn}\ X_n = \mathbf{B}_n$$

This process is called '*triangularization'*.

PART II: BACKSUBSTITUTION

From the triangular system of linear equations, first the value of x_n from the equation can be calculated as:

$$X_n = a_{n(n+1)} / a_{nn}$$

The value of $x_n$is substituted in other equations and then the rest values are calculated. This process is called *back-substitution.*

Example

Q: Solve the following system of linear equations using Gauss Elimination Method

$$2x_1 + 8x_2 + 2x_3 = 14$$
$$x_1 + 6x_2 - \ \ x_3 = 13$$
$$2x_1 - \ \ x_2 + 2x_3 = \ \ 5$$

Sol: In order to eliminate x_1 from the second and third equation, following transformation is applied:

$$R_2\text{-}(a_{21}/a_{11} * R_1) \quad = R_2\text{-}(1/2 * R_1)$$

The coefficients of the second equation are computed as:

$$a_{21}=a_{21}-1/2*a_{11}= 1-1/2*2 = 0$$
$$a_{22}=a_{22}-1/2*a_{12}= 6-1/2*8 = 2$$
$$a_{23}=a_{23}-1/2*a_{13}= -1-1/2*2 = -2$$
$$b_2=b_2-1/2*b_1= 13-1/2*14 = 6$$

Now apply transformation:

$$R_3\text{-}(a_{31}/a_{11} * R_1) \quad = R_3\text{-}(2/2 * R_1) \quad = R_3\text{-}R_1$$

The coefficients of the third equation are computed as:

$$a_{31}=a_{31}-a_{11}= 2-2 = 0$$
$$a_{32}=a_{32}-a_{12}= -1-8= -9$$
$$a_{33}=a_{33}-a_{13}= 2-2 = 0$$

$b_3 = b_3 - b_1 = 5 - 14 = -9$

Thus eliminating x_1 from the second and third equation, new system of linear equations is obtained:

$$2x_1 + 8x_2 + 2x_3 = 14$$
$$2x_2 - 2x_3 = 6$$
$$-9x_2 + 0x_3 = -9$$

In order to eliminate x_2 from the third equation, following transformation is applied:

$$R_3 - (a_{32}/a_{22} * R_2) \quad = R_3 - (-9/2 * R_2) \quad = R_3 + (9/2 * R_2)$$

The coefficients of the third equation are computed as:

$$a_{32} = a_{32} + 9/2 * a_{22} = -9 + 9/2 * 2 = 0$$
$$a_{33} = a_{33} + 9/2 * a_{23} = 0 + 9/2 * -2 = -9$$
$$b_3 = b_3 + 9/2 * b_2 = -9 + 9/2 * 6 = 18$$

Final system of linear equations is obtained:

$$2x_1 + 8x_2 + 2x_3 = 14$$
$$2x_2 - 2x_3 = 6$$
$$-9x_3 = 18$$

Through back-substitution, following solution values are obtained;

$x_3 = 18/-9 = -2$

$x_2 = (6 + 2x_3)/2 = 1$

$x_1 = (14 - 2x_3 - 8x_2) / 2 = 5$

2. GAUSS –JORDAN METHOD

The difference between the Gauss-Jordon and Gauss elimination is that in Gauss Jordon, the unknowns are eliminated from all other equations and not only from equations to follow, thus, removing the use of back-substitution process.

<u>Steps to Solve An Equation Using Gauss Jordon:</u>

Step 1: Eliminate x_1 from all equation except the first equation. This is done as follows:

Divide the first equation by a_{11}. Subtract from the second equation a_{21} times the first equation, subtract from the third equation a_{31} times the first equation and so on. Finally subtract from the n^{th} equation a_{n1} times the first equation.

Step 2: Eliminate x_2 from all equation except the second equation. This is done as follows:

Divide the second equation by a_{22}. Subtract from the second equation a_{12} times the second equation, subtract from the third equation a_{32} times the second equation and so on. Finally subtract from the n^{th} equation a_{n2} times the second equation.

The process will continue till x_n is eliminated from the first to $(n-1)^{th}$ equation. The final form of equation looks like:

$$x_1 + 0x_2 + 0x_3 + \ldots + 0x_n = b_1$$
$$0x_1 + x_2 + 0x_3 + \ldots + 0x_n = b_2$$
$$.$$
$$.$$
$$0x_1 + 0x_2 + 0x_3 + \ldots + x_n = b_n$$

The values of the unknowns are given by the coefficients on the right hand side of the equations.

Example

Q: Solve the following system of linear equations using Gauss Jordon Method.

$$2x_1 - 2x_2 + 5x_3 = 13$$
$$2x_1 + 3x_2 + 4x_3 = 20$$
$$3x_1 - x_2 + 3x_3 = 10$$

Sol: Step1 : Eliminate x_1 from all equations

a. Divide first equation by a_{11} i.e. 2

$$(2x_1 - 2x_2 + 5x_3 = 13)/2$$
$$= x_1 - x_2 + 2.5x_3 = 6.5 \text{------------------ eq.1}$$

b. Multiply this equation by a_{21} i.e. 2

$$= 2x_1 - 2x_2 + 5x_3 = 13$$

Subtracting from equation 2

$$(2x_1 + 3x_2 + 4x_3 = 20) - (2x_1 - 2x_2 + 5x_3 = 13)$$
$$= 5x_2 - x_3 = 7 \text{---------------------------eq. 2}$$

c. Multiply eq. 1 by a_{31} i.e. 3

$$= 3x_1 - 3x_2 + 7.5x_3 = 19.5$$

Subtracting from equation 3

$$(3x_1 - x_2 + 3x_3 = 10) - (3x_1 - 3x_2 + 7.5x_3 = 19.5)$$
$$= 2x_2 - 4.5x_3 = -9.5 \text{---------------------eq.3}$$

Our new set of equations is:

$$x_1 - x_2 + 2.5x_3 = 6.5$$

$5x_2 - x_3 = 7$

$2x_2 - 4.5x_3 = -9.5$

Step 2: Eliminate x_2 from all equations :

a. Divide second equation by a_{22} i.e. 5

$(5x_2 - x_3 = 7)/5$

$= x_2 – 0.2x_3 = 1.4$------------------ eq.2

b. Multiply this equation by a_{12} i.e. -1

$= - x_2 + 0.2x_3 = -1.4$

Subtracting from equation 2

$(x_1 - x_2 + 2.5x_3) –(- x_2 + 0.2x_3 = -1.4)$

$= x_1 +2.3x_3 = 7.9$-------------------eq. 1

c. Multiply eq.2 by a_{32} i.e. 2

$= 2x_2 – 0.4x_3 = 2.8$

Subtracting from equation 3

$(2x_2 - 4.5x_3 = -9.5) –(2x_2 – 0.4x_3 = 2.8)$

$= -4.1x_3 = -12.3$----------------------eq.3

Our new set of equations is:

$x_1 + 0x_{2+}\ 2.3x_3 = 7.9$

$x_2 – 0.2x_3 = 1.4$

$-4.1x_3 = -12.3$

Step3: Eliminate x_3 from all equations

a. Divide third equation by a_{33} i.e. -4.1

$(-4.1x_3 = -12.3)/-4.1$

$= x_3= 3$ ------------------ eq.3

b. Multiply this equation by 2.3

$= 2.3x_3= 6.9$

Subtracting from equation 1

$(x_1 + 0x_{2+}\ 2.3x_3 = 7.9) – (2.3x_3= 6.9)$

$= x_1 = 1$----------------------------eq.1

c. Multiply eq. 3 by a_{23} i.e. -0.2

$=- 0.2x_3= -0.6$

Subtracting from equation 2

$(x_2 – 0.2x_3 = 1.4) –(-0.2x_3= -0.6)$

$= x_2 = 2.0$----------------------eq.2

Our new set of equations is:

$x_1 = 1$

$x_2 = 2.0$

$x_3 = 3$

Thus, the required solution is:

$x_1 = 1, \quad x_2 = 2.0, \quad x_3 = 3$

ITERATIVE METHODS:

The iterative methods are preferred over direct methods particularly when the coefficient matrix is sparse i.e. have many zeros. These methods are more rapid and are more economical in memory requirements of a computer. These methods are also known as '*the methods of successive approximations*'. The necessary and sufficient condition for the use of these methods is that the diagonal elements of the coefficient matrix should be dominant. Such a system of linear equations is known as a *diagonal system*.

3. GAUSS –SEIDEL METHOD

It is an iterative method to solve the linear equations.

Steps to Solve An Equation Using Gauss Seidel:

Iteration 1:

a. Find the values of x_1 from the first equation by substituting the initial values of other unknowns.
b. Find the values of x_2 from the second equation by substituting the current values of x_1 and the initial values of other unknowns.
c. Find the values of x_3 from the third equation by substituting the current values of x_1 and x_2 and the initial values of other unknowns.

    - .
    - .
    - .

And so on, till the value of x_n is computed from the nth equation using current values of x_1, $x_2,\ldots,x_{n-1}$.

Iteration 2:

a. Find the values of x_1 from the first equation by substituting the values of other unknowns obtained in the first iteration.

b. Find the values of x_2 from the second equation by substituting the current values of other unknowns.
c. Find the values of x_3 from the third equation by substituting the current values of other unknowns.

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And so on, till the value of x_n is computed from the nth equation using current values of x_1, $x_2, \ldots, x_{n-1}$.

The iterative procedure is continued until the successive values of each unknown differ only within the permissible limits.

Example

Q: Solve the following system of linear equations using Gauss Seidel Method, correct to three decimal digits.

$$10x_1 + x_2 + 2x_3 = 44$$
$$2x_1 + 10x_2 + x_3 = 51$$
$$x_1 + 2x_2 + 10x_3 = 61$$

Sol: Since the system is diagonal system, therefore convergence is assured.

The given set of equations can be rewritten as:

$$x_1 = 1/10\ (44 - x_2 - 2\,x_3)$$
$$x_2 = 1/10\ (51 - 2x_2 - x_3)$$
$$x_3 = 1/10\ (61 - x_1 - 2\,x_2)$$

We start with initial approximation:

$$x_1 = x_2 = x_3 = 0$$

ITERATION 1: Substituting $x_2 = x_3 = 0$ in the first equation, we obtain

$$x_1 = 4.4$$

Substituting $x_1 = 4.4$ and $x_3 = 0$ in the second equation, we obtain

$$x_2 = 4.22$$

Substituting $x_1 = 4.4$ and $x_2 = 4.22$ in the third equation, we obtain

$$x_3 = 4.816$$

Therefore, the first set of approximation is:

$$x_1 = 4.4, \quad x_2 = 4.22, \quad x_3 = 4.816$$

ITERATION 2: Substituting $x_2 = 4.22$ and $x_3 = 4.816$ in the first equation, we obtain

$x_{1=}$ 4.0154

Substituting $x_{1=}$4.0154 and x_3=4.816 in the second equation, we obtain

$x_{2=}$ 3.0148

Substituting $x_{1=}$4.0154 and x_2=3.0148 in the third equation, we obtain

$x_{3=}$ 5.0955

Therefore, the second set of approximation is:

$x_{1=}$ 4.0154, $x_{2=}$ 3.0148, $x_{3=}$ 5.0955

ITERATION 2: Substituting $x_{2=}$ 4.22 and x_3=4.816 in the first equation, we obtain

$x_{1=}$ 4.0154

Substituting $x_{1=}$4.0154 and x_3=4.816 in the second equation, we obtain

$x_{2=}$ 3.0148

Substituting $x_{1=}$4.0154 and x_2=3.0148 in the third equation, we obtain

$x_{3=}$ 5.0955

Therefore, the second set of approximation is:

$x_{1=}$ 4.0154, $x_{2=}$ 3.0148, $x_{3=}$ 5.0955

ITERATION 3: Substituting $x_{2=}$ 3.0148and x_3=5.0955 in the first equation, we obtain

$x_{1=}$ 3.0794

Substituting $x_{1=}$3.0794and x_3=5.0955in the second equation, we obtain

$x_{2=}$ 3.9746

Substituting $x_{1=}$3.0794and x_2=3.9746 in the third equation, we obtain

$x_{3=}$ 4.9971

Therefore, the third set of approximation is:

$x_{1=}$ 3.0794, $x_{2=}$ 3.9746, $x_{3=}$ 4.9971

ITERATION 4: Substituting $x_{2=}$ 3.9746and x_3=4.9971 in the first equation, we obtain

$x_{1=}$ 3.0031

Substituting $x_{1=}$3.0031and x_3=4.9971 in the second equation, we obtain

$x_{2=}$ 3.9997

Substituting $x_{1=}$3.0031and x_2=3.9997 in the third equation, we obtain

$x_{3=}$ 4.8001

Therefore, the fourth set of approximation is:

$x_{1=}$ 3.0031, $x_{2=}$ 3.9997, $x_{3=}$ 4.8001

ITERATION 5: Substituting $x_{2=}$ 3.9997and x_3=4.8001in the first equation, we obtain

$x_{1=}$ 3.0400

Substituting $x_{1=}$ 3.0400 and x_3=4.8001in the second equation, we obtain

$x_{2=}$ 4.0120

Substituting $x_{1=}$ 3.0400and x_2=4.0120 in the third equation, we obtain

$x_{3=}$ 4.8360

Therefore, the fifth set of approximation is:

$x_{1=}$ 3.0400, $x_{2=}$ 4.0120, $x_{3=}$ 4.8360

ITERATION 6: Substituting $x_{2=}$ 4.0120and x_3=4.8360 in the first equation, we obtain

$x_{1=}$ 3.0316

Substituting $x_{1=}$ 3.0316 and x_3=4.8360 in the second equation, we obtain

$x_{2=}$ 4.0101

Substituting $x_{1=}$ 3.0316 and x_2=4.0101 in the third equation, we obtain

$x_{3=}$ 4. 9948

Therefore, the sixth set of approximation is:

$x_{1=}$ 3.0316, $x_{2=}$ 4.0101, $x_{3=}$ 4.9948

ITERATION 7: Substituting $x_{2=}$ 4.0101and x_3=4.9948 in the first equation, we obtain

$x_{1=}$ 3.0000

Substituting $x_{1=}$ 3.0000 and x_3=4.9948 in the second equation, we obtain

$x_{2=}$ 4.0001

Substituting $x_{1=}$ 3.0000 and x_2=4.0001 in the third equation, we obtain

$x_{3=}$ 5.0000

Therefore, the seventh set of approximation is:

$x_{1=}$ 3.0000, $x_{2=}$ 4.0001, $x_{3=}$ 5.0000

ITERATION 8: Substituting $x_{2=}$ 4.0001and x_3=5.0000 in the first equation, we obtain

$x_{1=}$ 3.0000

Substituting $x_{1=}$ 3.0000 and x_3=5.0000 in the second equation, we obtain

$x_{2=}$ 4.0000

Substituting $x_{1=}$ 3.0000 and x_2=4.0001 in the third equation, we obtain

$x_{3=}$ 5.0000

Therefore, the seventh set of approximation is:

$x_{1=}$ 3.0000, $x_{2=}$ 4.0000, $x_{3=}$ 5.0000

Comparing the approximations of the seventh and eighth iterations, there is no variation in the first four significant digits; therefore, we take the solution obtained at the end of eighth iteration as desired solution.

Therefore the solution correct to four significant digits is:

$$x_{1}= 3.0000, \qquad x_{2}= 4.0000, \qquad x_{3}= 5.0000$$

INTERPOLATION

Suppose x and y are two variables whose relation can be defined as:

$$Y=f(x) \qquad x_1<x<x_n$$

Where, x is an independent variable and y is dependent variable.

1. INTERPOLATION: Calculating / estimating value of dependent variable (y) from independent variable x, where $x_1<x<x_n$.
2. INVERSE INTERPOLATION: Calculating / estimating value of dependent variable (y) for given value of independent variable.
3. EXTRAOLATION: Estimating value of independent variable (y) for a given value 'x' outside in the range $x_1<x<x_n$

METHODS OF INTERPOLATION

There are a variety of methods available for interpolation, each having its own characteristic. The decision of using a particular method depends in tabulation of the function. They are classified into 2 types:

1. Methods for Equally Spaced Function 2. Methods for Unequally Spaced Function

1. METHODS FOR EQUALLY SPACED FUNCTION: include
   a. Newton's forward interpolation formula
   b. Newton's backward interpolation formula
   c. Gauss's formula
   d. Bessel's formula

   If the function is tabulated at equal intervals, then we can either use forward difference interpolation formula or backward difference interpolation formula. If the point to be interpolated lies in the upper half of table then the forward difference interpolation formula will give better approximation and if it lies in the lower half, backward difference interpolation formula will give better approximation.
2. METHODS FOR UNEQUALLY SPACED FUNCTION: include
   a. Lagrangian interpolation
   b. Newton's divided difference interpolation formula

LAGRANGIAN INTERPOLATION

If the function is tabulated at unequal intervals, then we use Lagrangian interpolation.

Consider a polynomial of form $y(x) = a_1(x-x_2)(x-x_3) + a_2(x-x_1)(x-x_3) + a_3(x-x_1)(x-x_2)$ passing through points (x_1,y_1), (x_2,y_2), (x_3,y_3) -------------------eq.1

At $x= x_1$ => $a_1= y_1/ ((x_1-x_2)(x_1-x_3))$
At $x= x_2$ => $a_2= y_2/ ((x_2-x_1)(x_2-x_3))$
At $x= x_3$ => $a_3= y_3/ ((x_3-x_1)(x_3-x_2))$
Substituting in eq.1
$Y(x) = y_1 * (x-x_2)(x-x_3)/ ((x_1-x_2)(x_1-x_3)) + y_2 * (x-x_1)(x-x_3)/ ((x_2-x_1)(x_2-x_3)) + y_3 * (x-x_1)(x-x_2)/ ((x_3-x_2)(x_3-x_2))$

The above equation for second order polynomial can be expressed as:

$$y(x)= \sum_{i=1}^{3} (yi) \prod_{j=1, j!=i}^{3} \frac{x - xj}{xi - xj}$$

In general, for ‘n’ points the expression can be represented as:

$$y(x)= \sum_{i=1}^{n} (yi) \prod_{j=1, j!=i}^{n} \frac{x - xj}{xi - xj}$$

and is known as Lagrangian polynomial.

Example

Q: Given the table of values as:

X	0	1	2	3
Y(x)	0	2	8	27

Find y (2.5).
Sol: Since there are 4 points, x=4
$Y(x) = y_1 * ((x-x_2)(x-x_3)(x-x_4))/ ((x_1-x_2)(x_1-x_3)(x_1-x_4)) + y_2 * (x-x_1)(x-x_3)(x-x_4)/ ((x_2-x_1)(x_2-x_3)(x_2-x_4)) + y_3 * (x-x_1)(x-x_2)(x-x_4)/ ((x_3-x_1)(x_3-x_2)(x_3-x_4)) + y_4 * (x-x_1)(x-x_2)(x-x_3)/ ((x_4-x_1)(x_4-x_2)(x_4-x_3))$

Substituting values, we get

$Y(2.5) = 0*((2.5-1)(2.5-2)(2.5-3))/((0-1)(0-2)(0-3)) + 2*((2.5-0)(2.5-2)(2.5-3))/((1-0)(1-2)(1-3)) + 8*((2.5-0)(2.5-1)(2.5-3))/((2-0)(2-1)(2-3)) + 27*((2.5-0)(2.5-1)(2.5-2))/((3-0)(3-1)(3-2))$

$= 15.313$

NEWTON'S METHOD OF INTERPOLATION:

Based on the type of difference being used, Newton's method of interpolation are divided into 3 categories, which are listed as:

1. Newton's forward interpolation formula.
2. Newton's backward interpolation formula.
3. Newton's divided difference interpolation formula.

If the function is tabulated at equal intervals, then we can either use forward difference interpolation formula or backward difference interpolation formula. If the point to be interpolated lies in the upper half of table, then the forward difference interpolation formula will give better approximation and if it lies in the lower half, backward difference interpolation formula will give better approximation. But if the function is tabulated at unequal intervals, then we can use the divided difference interpolation formula, which also works for equally spaced points.

NEWTON'S FORWARD DIFFERENCE INTERPOLATION FORMULA

Forward difference interpolation formula is used when the function is tabulated at equal intervals. If the point to be interpolated lies in the upper half of table, then the forward difference interpolation formula gives better approximation.

In order to interpolate at any point between x_k and $x_{k+1,}$ i.e. $x_k < x < x_{k+1}$, Newton's forward difference interpolation formula takes the form

$y(x) = y_k + \Delta y_k u + \Delta^2 y_k u(u-1)/2! + \ldots + \Delta^{n-k} y_k u(u-1)/(n-k)! \ldots (u – ((n-k) -1))$

where,

$\Delta y_k = y_{k+1} - y_k$

$\Delta^2 y_k = \Delta y_{k+1} - \Delta y_k$

$\Delta^3 y_k = \Delta^2 y_{k+1} - \Delta^2 y_k$

(Derivation of method to be done by student)

Example

Q: Given the table of values as:

X	2.0	2.25	2.50	2.75	3.0
Y(x)	9.00	10.06	11.25	12.56	14.00

Find y (2.35).

Sol:

	X	Y	$\Delta y_k = y_{k+1} - y_k$	$\Delta^2 y_k = \Delta y_{k+1} - \Delta y_k$	$\Delta^3 y_k = \Delta^2 y_{k+1} - \Delta^2 y_k$	$\Delta^4 y_k = \Delta^3 y_{k+1} - \Delta^3 y_k$
X1	2.0	9.00	10.06 - 9.00 =1.06	1.19 - 1.06 =0.13	0.12 - 0.13= -0.01	0.01-(-0.01) =0.02
X2	2.25	10.06	11.25 -10.06 =1.19	1.31 - 1.19 =0.12	0.13 - 0.12 = 0.01	
X3	2.50	11.25	12.56 -11.25 =1.31	1.44 - 1.31 =0.13		
X4	2.75	12.56	14.00 -12.56 =1.44			
X5	3.0	14.00				

Since 2.35 lies between x1 and x3, therefore we consider difference at second point.

$$y(x) = y_k + \Delta y_k u + \Delta^2 y_k u(u-1)/2! + \ldots + \Delta^{n-k} y_k u(u-1)/(n-k)! \ldots (u - ((n-k)-1))$$

Also, $u = (x-x_2)/h = (2.35 - 2.25)/0.25 = 0.4$

Y (2.35) =[10.06]+[1.19*0.4]+ ([0.12*0.4*(0.4-1)])/(2*1) + [(0.01*0.4*(0.4-1)*(0.4-2)]/(3*2*1)

=10.522

NEWTON'S BACKWARD DIFFERENCE INTERPOLATION FORMULA

Forward difference interpolation formula is used when the function is tabulated at equal intervals. If the point to be interpolated lies in lower half of table, then the backward difference interpolation formula gives better approximation.

In order to interpolate at any point between x_k and x_{k+1}, i.e. $x_k < x < x_{k+1}$, Newton's backward difference interpolation formula takes the form:

$$y(x) = y_k + \nabla y_k u + \nabla^2 y_k u(u+1)/2! + \ldots + \nabla^{k-1} y_k u(u+1)/(k-1)! \ldots (u + ((k-1) - 1))$$

(Derivation of method to be done by student)

Example

Q: Given the table of values as:

X	2.5	3.0	3.5	4.0	4.5
Y(X)	9.75	12.45	15.70	19.52	23.75

Find y (4.25).

Sol:

	X	Y(X)	∇y_i	$\nabla^2 y_i$	$\nabla^3 y_i$	$\nabla^4 y_i$
X1	2.5	9.75				
X2	3.0	12.45	12.45-9.75=2.70			
X3	3.5	15.70	15.70-12.45=3.25	3.25-2.70=0.55		
X4	4.0	19.52	19.52-15.70=3.82	3.82-3.25=0.57	0.57-0.55=0.02	
X5	4.5	23.75	23.75-19.52= 4.23	4.23-3.82=0.41	0.41-0.57=-0.16	-0.16-0.02=-0.18

Since 4.25 lies between x4and x5, therefore we consider difference at fifth point.

$$y(x) = y_k + \nabla y_k u + \nabla^2 y_k u(u+1)/2! + \dots + \nabla^{k-1} y_k u(u+1)/(k-1)! \dots (u + ((k-1)-1))$$

Also, u= $(x-x_5)/h$ = (4.25 -4.5)/0.5 =-0.5

Y (4.25) = [23.75] + [4.23*-0.5] + ([0.41*-0.5*(-0.5+1)])/(2*1) + [(0.16*-0.5*(-0.5+1) * (-0.5+2)]/(3*2*1)

= 21.601

EXERCISE

Q: Derivation of Newton's forward interpolation method and Newton's backward interpolation method.

Q: Programmatic implementation of all the methods in C or C++.